

Advanced Volatility

Add a little wind and we get a little increase in volatility. Add a hurricane and we get a huge increase in volatility.



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About the book

Reading about this material is essential. The book will add a lot of explanatory power to my lectures.

You must read Hull's material on complex volatilities in chapter 21.

Alas and unhappily, we do not have time to cover the material on Maximum Likelihood Estimation methods and you are not accountable for that material.

Although you will not be required to memorize the volatility estimates shown here, you will be required to *recognize them and explain their features.*

What we can't forget

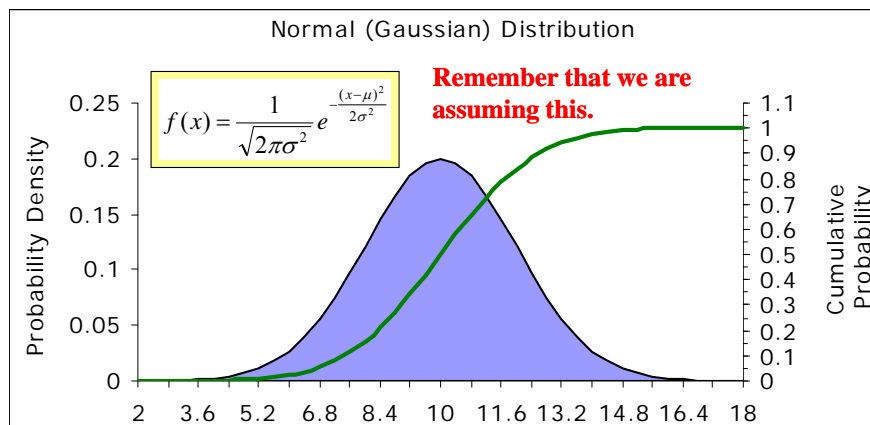
As we proceed in this discussion of volatility estimates and modeling of risk, we never abandon the assumption that the determination of financial asset prices are *economic*. That is,

$$P = f(X, Y, Z, X_e, Y_e, Z_e, X_{t-n}, X_{e_{t-n}}, \text{etc.})$$

where X, Y, and Z are meant to represent the vector (sometimes large) of all variables, like interest rates, inflation rates, income rates, corporate earnings rates, and so forth, that have an effect upon price, the "e" subscript represents *expectations* of the same upon price, and "t-n" represents *time lag structures* of these same variables.

Our modeling of this, whether implicit or real, often also assumes properties like continuity, monotonicity and sometimes assumes that the modeled relationship, as represented mathematically. doesn't change (much) over time. But we know there are problems with this.

The assumed probability distribution of FA continuous growth rates (or deltas)



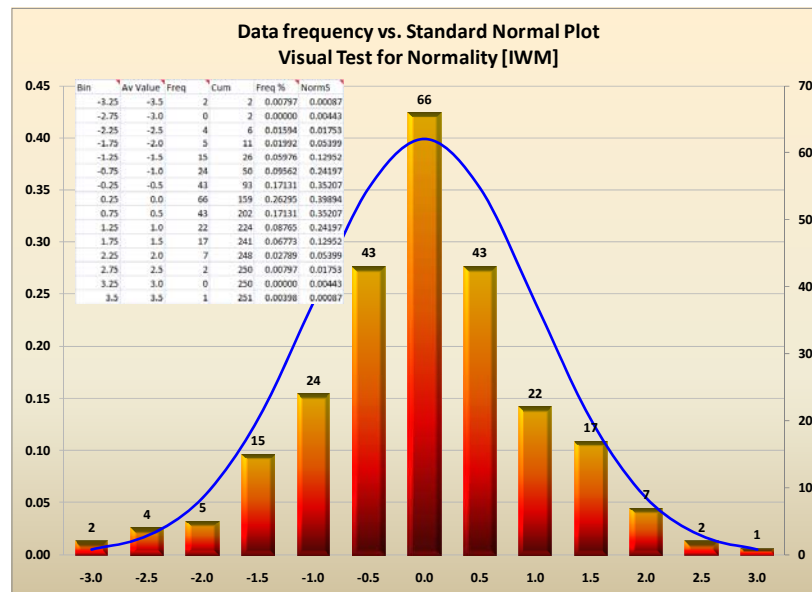
... and you should test for normalcy using one (at least) of the standard tests, like Anderson-Darling.

My approach to normality for ETFs ...

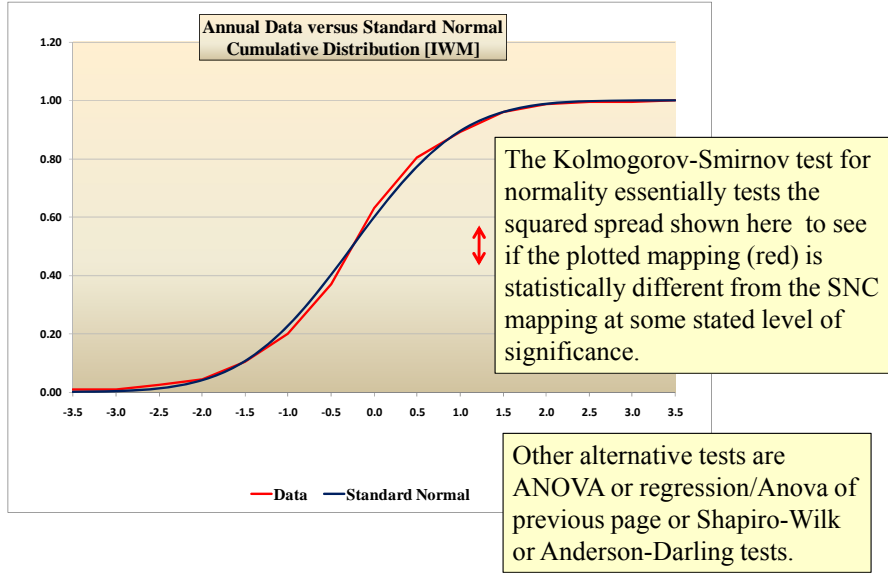
(Example: IWM) Using one year of daily adjusted close and volume data, use a standard worksheet to calculate DCGR and these stats for one year, 60 days, and 30 days from the same data set, and convert all data to norms built around one standard deviation (to compare to SN distribution of the same dispersion). This also allows you to spot outliers (look at Min and Max DCGR numbers).

Duration	
One Year:	
Average DGR:	0.00065
Standard Deviation:	0.01563
Average ABS DCGR:	0.01161
Min:	-0.05204
MaX:	0.05460
Min Norm DCGR:	-3.37101
Max Norm DCGR:	3.45076
<F9> to calculate	
60 day:	
Average DGR:	0.00009
Standard Deviation:	0.01154
Average ABS DCGR:	0.00874
Min:	-0.02604
MaX:	0.02289
Min Norm DCGR:	-2.26513
Max Norm DCGR:	1.97593
30 Day:	
Average DGR:	-0.00043
Standard Deviation:	0.01236
Average ABS DCGR:	0.00951
Min:	-0.02604
MaX:	0.02289
Min Norm DCGR:	-2.07266
Max Norm DCGR:	1.88815

... which allows a visual normality check:

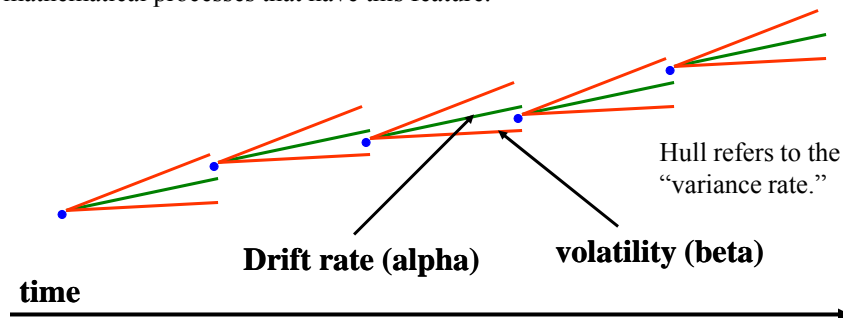


... and a cum dist check as well



About drift and volatility

We are going to regard the path of securities (and their derivatives) prices as a Markov Process with actual price behavior over time reflecting *drift* and *volatility*, where the latter is represented by a Gaussian distribution. The resulting pattern will reflect randomness with a trend. We are searching for mathematical processes that have this feature.



Classical Historical Volatility

Using closing daily data for, say, a year:

$$\mu_t = \ln(C_t / C_{t-1})$$

$$\bar{\mu} = \frac{1}{T} \sum_{t=1}^T \mu_t$$

$$\sigma_c = \sqrt{\frac{1}{T} \sum_{t=1}^T (\mu_t - \bar{\mu})^2}$$

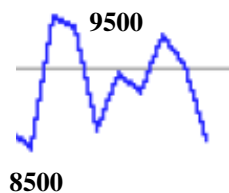
Note that this is a specific application of using standard deviation for volatility that we have discussed all along.

Because this is drawn from a population, some would use $T-1$.

What data frequency should we use?

Even when estimating weekly or monthly volatility over a span of years, you should try to use daily data and then use a formula to convert daily volatility to a larger aggregate.

Why? Because if you, for example, use monthly data to estimate monthly volatility, your arbitrary choice of dates may mask interim volatility.



This is the DJIA in Oct 08. The endpoints are at more or less the same value (8500) But look at the volatility in between. In fact, this was the most volatile month in stock market history.

SD Volatility Time Conversions

When converting from standard deviation daily volatility to larger intervals, you take the daily standard deviation and multiply it times the square root of the number of days in the larger time interval. However, if there are, for example, only five trading days in a week (hence only five observations) and only 252 trading days in a year, then the number of days must be 5 and 252, not 7 and 365.

However, if you have a daily volatility measure for a stock and you are basing an option trade upon that measure, and the option expires in **one calendar month**, then the relevant multiplier for that trade is the square root of 30, not 22.

$$\sigma_{weekly} = \sigma_{daily} \sqrt{5} = 2.24\sigma_{daily}$$

$$\sigma_{monthly} = \sigma_{daily} \sqrt{22} = 4.69\sigma_{daily}$$

$$\sigma_{annual} = \sigma_{daily} \sqrt{252} = 15.9\sigma_{daily}$$

The Sharpe Ratio

Historical Sharpe Ratio: The ratio of the stock's (or other FA) historical rate of return over it's volatility, over the same period:

$$SR = \frac{\mu}{\sigma}$$

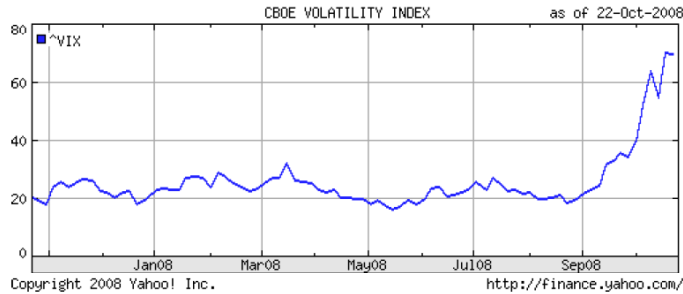
Some versions make calculate this as an opportunity-cost return by subtracting the risk-free interest rate from yield:

$$SR = \frac{(\mu - r)}{\sigma}$$

Investment strategists may replace mu with their alpha and historical standard deviation with implied volatility.

Dynamic/Variable Volatility: Wrestling with Time

The classical approach is useful for some applications But daily volatility is not a constant! Look at the VIX:

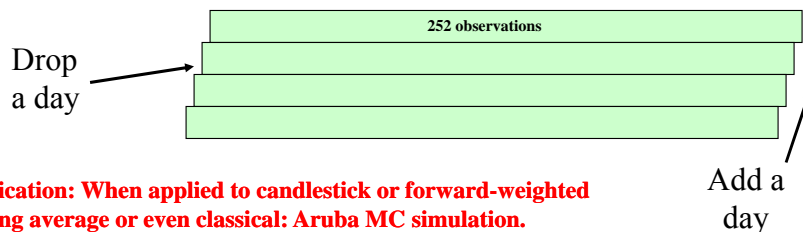


For example, if you trade options you would be out of your mind to use a constant historical volatility assumption.

Therefore we need a model that allows for variable volatility or dynamic volatility, or both.

Dynamic Daily Volatility Models

The dynamic model, which almost always concerns itself with daily volatility, given some timed data set, like one year, adds one new observation each day and drops the oldest observation, recalculating the volatility and treating the volatility as non-constant over time.



Application: When applied to candlestick or forward-weighted moving average or even classical: Aruba MC simulation.

Building a dynamic model: the start

From Hull, page 461: "Define σ_n as the volatility of a market variable on day n , as estimated at the end of day $n-1$. The square of the volatility, σ_n^2 , on day n is the *variance rate*." Assume also that we are using m observations (say 252) and that we will undertake a dynamic or rolling estimate every day, dropping the oldest observation and adding the most recent. Then

$$\mu_i = \ln \frac{C_i}{C_{i-1}}$$

$$\bar{\mu} = \frac{1}{m} \sum_{i=1}^m \mu_{n-i}$$

$$\sigma_n^2 = \frac{1}{m-1} \sum_{i=1}^m (\mu_{n-i} - \bar{\mu})^2$$

Note the notation. Also note that this gives equal weight to all observations of μ . We want to change that.

Another step: calculating daily volatility (shortcut)

(See Hull's explanation for this, equation 21.2 and 21.3): When monitoring daily activity we can assume μ to be zero because it is usually close to zero (see Hull's footnote 2) and because we can add any drift back in later. Now we redefine the daily change in the close to a discrete change and recalculate variance using the following formulas:

$$\mu_i = \ln \left(\frac{C_i}{C_{i-1}} \right)$$

$$\mu_i = \frac{C_i - C_{i-1}}{C_{i-1}}$$

$$\sigma_n^2 = \frac{1}{m} \sum_{i=1}^m \mu_{n-i}^2$$

This is a new variance rate and the SQRT is volatility.

Note that this is a weighted sum where each weight has the value $1/m$. Now we want to change the weights.

Adding weights weighted toward the present

With dynamic and variable volatility, we have good reason to believe that our weights should be changed from a constant to weights that give more importance to recent data. To do this, we can change our volatility estimate to a modification of the formula on the previous page,

$$\sigma_n^2 = \sum_{i=1}^m \alpha_i \mu_{n-i}^2 \quad \sum_{i=1}^m \alpha_i = 1$$

These weights might have values like 0.7, 0.2, and 0.1. Again, volatility is the SQRT of the term above.

Exponential Weighting

$$\sum_{t=1}^n e^{\alpha t} = 1$$

Where alpha will have a value of around -0.68.

Exponential Weights 5 & 10 periods

alpha5						
Time	1	2	3	4	5	Sum
	0.5086	0.2587	0.1316	0.0669	0.0340	1.0000

alpha10						
Time	1	2	3	4	5	
	0.5003	0.2503	0.1252	0.0627	0.0314	
Time	6	7	8	9	10	Sum
	0.0157	0.0078	0.0039	0.0020	0.0010	1.0003

EWMA weights

When measuring daily volatility assuming the growth rate to be zero, then the daily measure of volatility that is EWMA weighted is determined by

$$\sigma = \sqrt{\sum \alpha_t \mu_t^2}$$

This is a variation of our old formula for a rolling variance:

$$\sigma_n^2 = \frac{1}{m} \sum_{i=1}^m \mu_{n-i}^2$$

(Or we could use the ARCH/GARCH model with mean reversion, where the alphas would be done the same way)

The ARCH model

Remember historical variance? In this model we introduce historical variance back into the model with the equation below. In effect, we are weighting long-term variance with short-term dynamic variance.

$$\sigma_n^2 = \mathcal{W}_L + \sum_{i=1}^m \alpha_i \mu_{n-i}^2$$

$$\gamma + \sum_{i=1}^m \alpha_i = 1$$

Again , volatility is the SQRT of this,

Application: Writing covered calls.

The EWMA model

The popular exponentially weighted moving average model uses weights that decrease exponentially as we move back in time. This is satisfied by the condition

$$\alpha_{i+1} = \lambda \alpha_i \quad 0 < \lambda < 1$$

This seems like it would result in a cumbersome equation but results in a geometric series mathematically reducible to the practical expression (see Hull section 21.2 for the reduction):

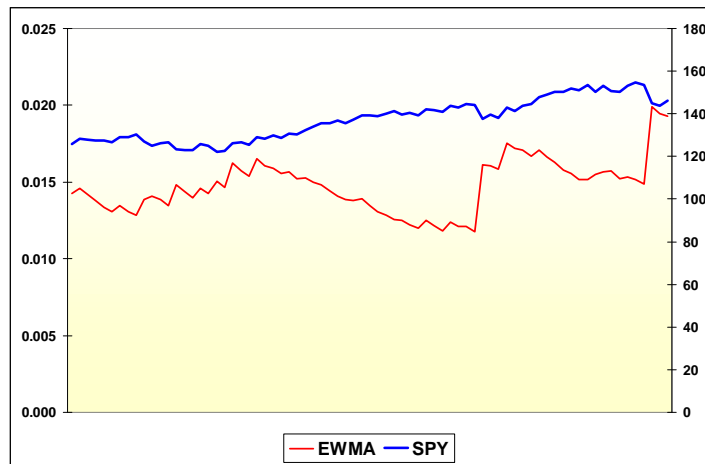
$$\sigma_n^2 = \lambda \sigma_{n-1}^2 + (1 - \lambda) \mu_{n-1}^2$$

Volatility is the square root of this.

*relevance to the Aruba model.

This approach requires that relatively little data be stored.* A J.P. Morgan team found that 0.94 worked well for (see Hull, p. 480). Also see numerical example 21.1 in Hull.

SPY and EWMA Weekly Volatility February 06 to August 07



The Garch(1,1) Model

The Garch(1,1) model is a blend between the EWMA model and the ARCH model. It combines the reduced exponentially weighted moving average with the long-range variance rate in the ARCH model.

$$\sigma_n^2 = \gamma V_L + \alpha \mu_{n-1}^2 + \beta \sigma_{n-1}^2$$

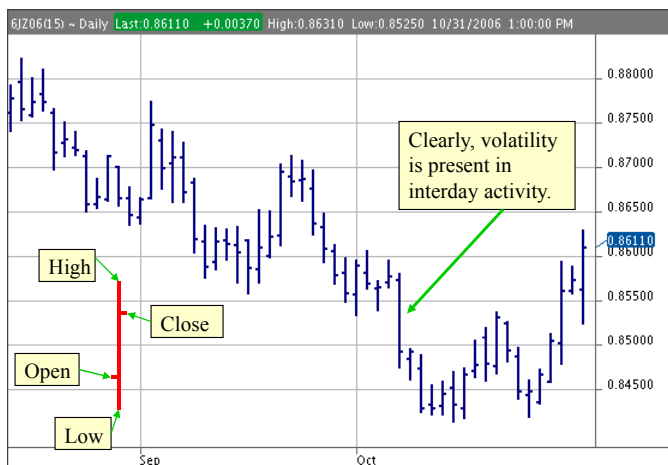
$$\alpha + \beta + \gamma = 1$$

This model has a mean reversion property* that over time the variance is pulled back toward the long run average represented by V_L . (See text).

This model requires the estimation of parameters as discussed in Hull in Section 21.3 and using the Maximum Likelihood Method described in 21.12. I'm not teaching this nor require you to know it but remember that this application is in the book.

*Useful for GE but not DNDN.

Candlestick data: O,H,L,C



Most historical data series for financial market prices include the open, high, low, and close, and charts are often offered with this data, as is shown here for this Yen CME futures graph.

Daily (Candlestick) estimator

Rogers and Satchell (1991) estimator (note that this estimator is independent of drift):

$$V = \frac{1}{T} \sum_{i=1}^T \left(\ln(H_i/C_i) \ln(H_i/O_i) + \ln(L_i/C_i) \ln(L_i/O_i) \right)$$

For example, if the candlestick values for H,C,O,L are 36,33,32 and 30, then

$$V = \frac{1}{T} \sum_{i=1}^T \left(\ln(36/33) \ln(36/32) + \ln(30/33) \ln(30/32) \right)$$

Application: Strangles, straddles, and straight option trades.

Problem with the Candlestick estimator

Candlestick (or weighted dynamic candlestick) is useful for evaluating volatility for options trades because it captures intra-day volatility. **But** it doesn't capture after-hours or inter-day volatility and that is a problem. Much of the volatility happens after hours and especially over the weekend.



Thanks to Jason
Christianson and
Asaf Bernstein '08

You either have to go back to close-to-close volatility, but dynamically weighted (as shown in next slide sequence) or try something like Open-to-Close then Close-to-Open, treating the market open as a half day and after hours as a half day.

What I like to do for options trades (in addition to what I said at opening)

- First look at the VIX and its graph over the last year. I never take my eye off the VIX.
- Do all of the calculations and checks discussed at opening.
- Decide what to do if it doesn't fit very well ...
 - figure out the outliers and what to do about them if a problem
 - if a terrible unfixable fit, stop using a statistical approach
- Don't be afraid to look at the graphs to see what they seem to say.
- Compare the different duration volatility estimates and figure out why they are different if considerably different.
- Maybe do EWMA weights (can see it in the data typically)
- Calculate the implied daily volatility (not taught yet) for this option in question and others in the chain if possible and of course compare to the historical calculation (the comparison is often the basis of a specific strategy).
- Given the strategy in question, do modeling sensitivity analysis.

The CBOE VIX Volatility formula

$$\sigma^2 = \frac{2}{T} \sum_i \frac{\Delta K_i}{K_i^2} e^{RT} Q(K_i) - \frac{1}{T} \left[\frac{F}{K_0} - 1 \right]^2$$

σ = VIX/100
 T = Time to expiration
 F = Forward index level derived from index option prices
 K_0 = First strike price below F
 K_i = Strike price of i^{th} out of the money option,
 ΔK_i = Mid point between strike prices
 Q = Mid point of bid/ask spread for each strike price
 R = Risk-free interest rate

See the online document: *The CBOE Volatility Index – VIX* on the CBOE site.

2-asset Portfolio Variance Sums

Variance is purely additive if two variables are strictly independent:

$$V(x + y) = V(x) + V(y) + 2COV(x, y)$$

remembering that Covariance is equal to the Correlation Coefficient (0 if no perfectly independent, 1 if perfectly correlated, -1 if perfectly polar) times the product of the standard deviations:

$$COV(x, y) = CORREL(x, y) \times SD(x) \times SD(y)$$

Simple example of diversification using our formula:

Suppose you have two uncorrelated stocks, $X(\mu, \sigma)$ $X_1(0.02, 0.03)$ and $X_2(0.04, 0.05)$. If you are risk-averse, you may want to put all of your money in stock X_1 and accept the lower 2% yield. But what if you split your portfolio 50/50, giving you a 3% yield? What would your risk be??

$V_1 = 0.0009$ and $V_2 = 0.0025$ and each alpha equals 1/2. Therefore $V_{1,2} = 0.25 \times (0.0009 + 0.0025) = 0.00085$ and $0.00085^{1/2} = 0.0291$.

Therefore, by diversifying your portfolio you have raised your yield by 50% while lowering your risk.

Stat-Arb and Portfolio Volatility

When building portfolios, such as for mutual funds and hedge funds, then the volatility of the entire portfolio will matter a great deal.

One clear hedge fund objective is to raise, or try to maximize, the **Sharpe Ratio** of the portfolio:

$$\max SR = \frac{\mu}{\sigma}$$

For a stat-arb portfolio, the goal is to make the portfolio market-neutral which implies a covariance with a market index like the S&P500 as zero.

n-asset Portfolio Volatility

If we have 'n' assets in the portfolio, then we calculate the variance using this additive formula:

$$VAR \left[\sum_{i=1}^n x_i \right] = \sum_{i=1}^n VAR(x_i) + 2 \sum_{i=1}^{n-1} \sum_{j=(i+1)}^n COV(x_i, x_j)$$

which is easy to program *if** you have the data. What finally matters, of course, is the square root of this term, the **standard deviation**, which is our volatility measure.

*this requires the calculation of all 'n' standard deviations and all $\sum_{i=1}^{n-1} i$ paired correlations (15 for 6 stocks).

For reference and discussion, see <http://mathworld.wolfram.com/Variance.html>

Weight-adjusted n-asset Portfolio Volatility

If you assign weights to your portfolio, represented here as alphas, which of course you would, then the variance formula is:

$$VAR\left[\sum_{i=1}^n \alpha_i x_i\right] = \sum_{i=1}^n \alpha_i^2 VAR(x_i) + 2\sum_{i=1}^{n-1} \sum_{j=(i-1)}^n \alpha_i \alpha_j COV(x_i, x_j)$$

The volatility of this portfolio, the standard deviation, is the square root of this expression. Clearly, the greater the independence of your portfolio components, the smaller the risk. This shows the benefits of diversification into *non-correlated* stocks.

Help from David
Coates '08

Coding the Covariance (prior 2 equations)

For the covariance part of the equation only,

```
for I = 1 to (n-1) do
  for J = (i+1) to n do
```

```
  COV(I,J) = CORREL(I,J)*SD(I)*SD(J);
  SUMCOV = SUMCOV + COV(I,J);
```

```
end;
end;
```

```
SUMCOV = 2*SUMCOV;
```

and the weighted portfolio calculation would be the same except

```
WCOV(I,J) = a(I)*a(J)*CORREL(I,J)*SD(I)*SD(J);
```

$$\begin{bmatrix} S_1 & S_2 & S_3 & S_4 \end{bmatrix} \begin{bmatrix} 0 & C_{12} & C_{13} & C_{14} \\ 0 & 0 & C_{23} & C_{24} \\ 0 & 0 & 0 & C_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix} \times 2$$

Memo slide for sticklers for accuracy (a desirable trait), those of you who want to work in finance, and you coders who own a laptop and want to retire before age 35 trading off of any beach with a wireless setup.

Summary of Applications

- **Historical:** Measures long-term volatility from established data set, Beta-like calculations, used as base for ARCH and Garch models.
- **Candlestick:** Typically daily or less (hourly) volatility for volatile markets like FOREX.
- **Dynamic:** Short-term (daily, weekly) rolling volatility for stocks, options, indexes, yields.
- **Portfolio:** For building portfolios with low-risk objectives, like stat-arb or Sharpe max.

Jan 2009 Crude Contract (2008)



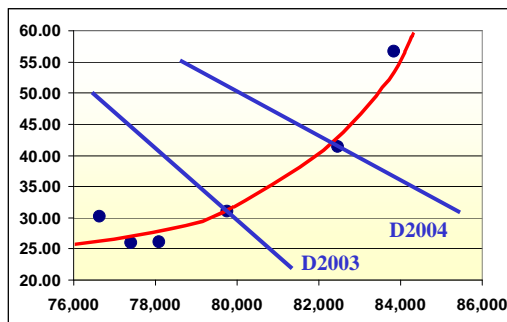
The ingredients of an explanation ...

- Oil probably has a **price supply elasticity** of about 0.25 (inelastic) which implies a severe price reaction (4X%) to fluctuations in global oil demand.
- The global economy went from **robust expansion to severe contraction** in one year, which would explain a severe expansion, then contraction, in the price of oil.
 - ... but frankly, in the ranges of \$40 to \$140???
- With this kind of momentum movement in first one direction, then the opposite, one can expect **volatility** to rise, maybe to a multiple of its original value (we certainly saw that in stocks).
- So ...

Severe Supply Inelasticity of Oil

Global Oil Market		
	Demand	Price
2000	76,619	30.26
2001	77,406	25.95
2002	78,082	26.15
2003	79,742	30.99
2004	82,452	41.47
2005	83,837	56.70

Daily millions barrels
Source: EIA

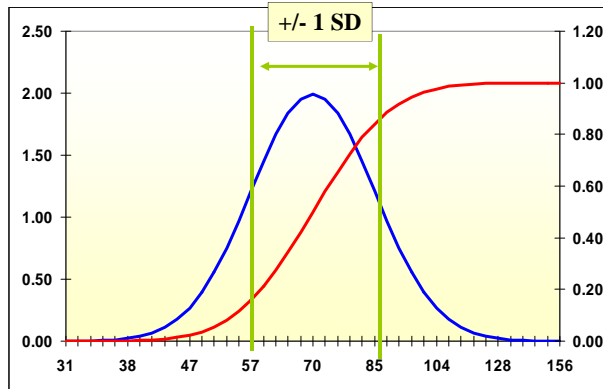


WEFA study commissioned by the American Petroleum Institute in 1990 concluded that the supply elasticity of oil was only 0.13! (controversial)

Data source: Energy Information Agency.

If oil at \$70 has an annual volatility of 0.20 ...

Value	Cumulative Probs
46.9224	-2 0.0228
57.3112	-1 0.1587
70.0000	0 0.5000
85.4982	1 0.8413
104.4277	2 0.9772

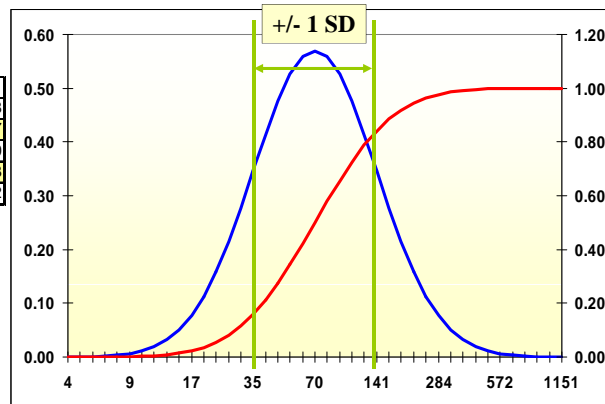


The mapping shows the range of continuous growth rates for the price of oil, which is normal. The abscissa is transformed to the actual possible prices of oil, and is therefore log-normal. Look at the 1-sigma range of prices.

... but if that volatility balloons to 0.70

Value	Cumulative Probs
17.2618	-2 0.0228
34.7610	-1 0.1587
70.0000	0 0.5000
140.9627	1 0.8413
283.8640	2 0.9772

A move from an annual volatility of 20% to 70% implies a move in the daily, observable volatility from 1% to 3.5%.



... the 1-sigma range for oil runs from \$35 to \$140. Look familiar?? Is this unexplainable or mysterious? No.