Working on this lecture while at Aqua Caliente. The view from outside the window.

The Greeks

... and implied volatility

Implied Volatility

Our theory suggests that the price of a call will be a function of the price of the stock, the strike price of the call, the volatility of the stock, and the time to expiration:

$$P_c = f\left(P_s, SP, \sigma_s, t\right)$$

Therefore we can conclude that if certain mathematical conditions are met, we can restructure this so that volatility is the dependent variable:

$$\sigma_s = g\left(P_c, P_s, SP, t\right)$$

When these functions take explicit form, as they do in the Black-Scholes and other options pricing models, then the solution value for volatility is called ***implied volatility***.

… note

2012 Nissan GT-R: 3.8L twin turbocharged 24-valve V6, 530 HP, 448 ft-lb torgue, ATTESA E-TS AWD, yaw-based torque distribution.

These Greeks all have formulas that emerge from options pricing models like Black-Scholes.

These formulas are in chapter 17.

They are not derived here – just shown.

The Delta and Gamma

$$\Delta = \frac{\partial P_c}{\partial P_s}$$

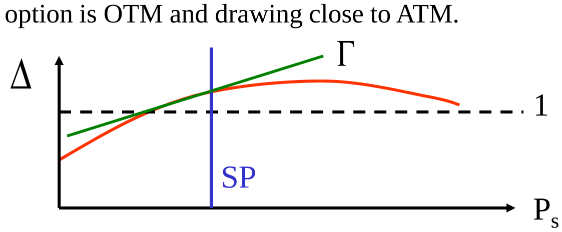
$$\Gamma = \frac{\partial^2 P_c}{\partial P_s^2}$$

The delta is defined to be the spot change in the price of a call (or put) relative to the change in the price of the underlying stock.

The delta is not a constant. For a call option, its mapping will loosely follow the diagram below. For that reason we also want to define the gamma, which measures the rate of change of the delta. Generally, the delta is below 1 for an OTM call option, converges to a value slightly more than 1 ATM, then converges back to 1 as the stock goes deeply ITM.

This implies that gamma is positive and falling when the option is OTM and drawing close to ATM.

Note: The mapping to the left of SP might be quasi-concave, with a rising then falling gamma.



Mudd Finance

Calculation of the Delta and Gamma

$$\Delta_c = N(d_1)$$

$$\Delta_p = N(-d_1)$$

$$\Gamma = \frac{\left(\frac{1}{\sqrt{2\pi}} e^{\frac{-d_1^2}{2}} \right)}{P_0 \sigma \sqrt{T}}$$

See chapter 17 of Hull.

The Gamma formula treats time as one year so would have to be converted if using daily volatility.

N is the value of the cumulative standard normal density function up to *d* and d_1 and d_2 have the same meaning as in previous lectures.

Mudd Finance

Delta Hedging

The N(-d1) in our option pricing model is the **delta**. According to the real IWM put pricing shown here, when IWM was 82.09, the IDV 0.0107, and the put at 1.06, the **delta = 0.3297**. That implies that if the stock falls by 3 cents, the put should rise by a penny, which is confirmed by the sensitivity measure of the far right.

Put Option Price Calculator
Daily Volatility

Stock symbol:	IWM
Put option:	May
Date Today:	4/19/2011
Expiration Date:	5/21/2011
DTM:	32
Stock Price:	82.09
Strike Price:	80.00
Daily Volatility:	0.0107
Interest Rate:	0.010
Time:	32
d1 Numerator:	0.02667
Duration Volatility:	0.06053
Delta N(-d1):	0.3297
N(-d2):	0.3519
Option Price:	1.06
Option Premium:	1.06

Put Option Price Calculator
Daily Volatility

Stock symbol:	IWM
Put option:	May
Date Today:	4/19/2011
Expiration Date:	5/21/2011
DTM:	32
Stock Price:	82.06
Strike Price:	80.00
Daily Volatility:	0.0107
Interest Rate:	0.010
Time:	32
d1 Numerator:	0.02631
Duration Volatility:	0.06053
Delta N(-d1):	0.3319
N(-d2):	0.3542
Option Price:	1.07
Option Premium:	1.07

If therefore you wanted a short-term "perfect hedge" (like a weekend hedge) you would buy **(1/delta) = 3** puts for every share you own. For example, you would hedge a 1,000 share portfolio worth $82,090 with 3,000 puts worth $3,180 (keep in mind that you can sell the puts on Monday – this is not net insurance cost).

Limitations of Delta Hedging and calculating the insurance cost

Put Option Price Calculator (Daily Volatility)	1	2	3
Stock symbol:	IWM	IWM	IWM
Put option:	May	May	May
Date Today:	4/19/2011	4/19/2011	4/19/2011
Expiration Date:	5/21/2011	5/21/2011	5/21/2011
DTM:	32	32	32
Stock Price:	82.09	81.09	81.09
Strike Price:	80.00	80.00	80.00
Daily Volatility:	0.0107	0.0107	0.0107
Interest Rate:	0.010	0.010	0.010
Time:	32	32	29
d1 Numerator:	0.02667	0.01441	0.01433
Duration Volatility:	0.06053	0.06053	0.05762
Delta N(-d1):	0.3297	0.4059	0.4018
N(-d2):	0.3519	0.4295	0.4242
Option Price:	1.06	1.42	1.33
Option Premium:	1.06	1.42	1.33

(1-2) Because the gamma is not zero, the delta is biased for larger moves like $1 in this case – the implied delta was 0.36, which suggests that the delta overestimated the size of our put position. (3), which reflects 3 days of time decay, shows that our insurance was rather expensive (9 cents per share).

Hedging by writing a DITM covered call

Call Option Price Calculator (Daily Volatility)	1	2	3
Stock symbol:	IWM	IWM	IWM
Call option:	May	May	May
Date Today:	4/19/2011	4/19/2011	4/19/2011
Expiration Date:	5/21/2011	5/21/2011	5/21/2011
DTM:	32	32	32
Stock Price:	82.11	78.11	78.11
Strike Price:	73.00	73.00	73.00
Daily Volatility:	0.0129	0.0129	0.0129
Interest Rate:	0.010	0.010	0.010
Time:	32	32	29
d1 Numerator:	0.11848	0.06854	0.06846
Duration Volatility:	0.07297	0.07297	0.06947
Delta N(d1):	0.9478	0.8262	0.8378
N(d2):	0.9395	0.8068	0.8202
Option Price:	9.30	5.69	5.62
Option Premium:	0.19	0.58	0.51

Too expensive!! Why? Look at the change in delta ***and*** the rise in option premium with no change in volatility.

Loss on stock:	4.00
Gain on call (no time 1-2):	3.61
Gain on call (3 day 1-3):	3.68
Net (3 day):	-0.32

Hedging by writing an OTM covered call

Mudd Finance

Here the delta suggests that we hedge 2.3 to 1, but a major move will leave us vulnerable because obviously the gain on the call cannot exceed its value.

Call Option Price Calculator (Daily Volatility) — 1

Stock symbol:	IWM
Call option:	May
Date Today:	4/19/2011
Expiration Date:	5/21/2011
DTM:	32
Stock Price:	82.10
Strike Price:	83.00
Daily Volatility:	0.0088
Interest Rate:	0.010
Time:	32
d1 Numerator:	-0.01002
Duration Volatility:	0.04978
Delta N(d1):	0.4202
N(d2):	0.4009
Option Price:	1.26
Option Premium:	1.26

Call Option Price Calculator (Daily Volatility) — 2

Stock symbol:	IWM
Call option:	May
Date Today:	4/19/2011
Expiration Date:	5/21/2011
DTM:	32
Stock Price:	78.10
Strike Price:	83.00
Daily Volatility:	0.0088
Interest Rate:	0.010
Time:	29
d1 Numerator:	-0.06005
Duration Volatility:	0.04739
Delta N(d1):	0.1025
N(d2):	0.0943
Option Price:	0.19
Option Premium:	0.19

Loss on stock:	4.00
Gain on call (3 day 1-2):	1.06
Call multiple:	2.30
Net (3 day):	-1.56

The Theta

Mudd Finance

$$\Theta = \frac{\partial P_c}{\partial t}$$

The theta captures the sensitivity of the price of the option to elapsed time as the option approaches maturity. The theta more or less represents the "time decay" of an option.

Theta is represented as a negative number, because it is meant to measure how much the value of the option will decline if time passes *and nothing else happens*, such as a change in the value of the stock or a change in the stock's volatility.

If you look at Hull's equation 17.2. Theta for a call (see book for a put) can be expressed as

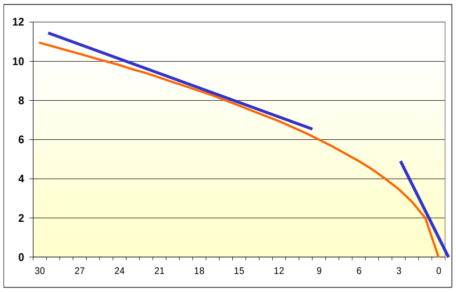
$$\Theta_c = \frac{P_0\left(\frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}\right)\sigma}{2\sqrt{t}} - rKe^{-rt} \times N(d_2)$$

This implies that the theta for a call option with a short duration will be greater (in absolute value) than one with a longer duration, i.e. more sensitive to the effect of one passing day if nothing changes.

Mudd Finance

Theta and Time Decay

(atm call option)



Theta is not time decay, it represents the rate of time decay. As such, the absolute value of theta rises, representing a quickly deteriorating time decay in the days just prior to expiration.

Shown here is the characteristic time decay of an atm call option that stays atm throughout its life. The ordinate axis is a relative measure of the ***time premium*** of the option.

Generally theta (absolute value) rises slightly in the first 20 days, producing a near-linear decay. Then in the final week theta becomes much larger and the time decay become pronounced.

Mudd Finance

The Vega

$$V = \frac{\partial P_c}{\partial \sigma_s}$$

The vega is the all-important sensitivity of the price of the option to the change in volatility. Clearly this will be an important positive number. It should be understood that the variance term represented here is daily volatility of the stock, even though the relevant variance one must consider when entering an option is the volatility of the time period remaining in the option's life (which will equal this times the square root of time).

$$V = P_0 \sqrt{\mathrm{T}} \left(\frac{1}{\sqrt{2\pi}} e^{-d_1^2/2} \right)$$

Mudd Finance

The Rho (interest rate sensitivity)

$$Rho_c = KTe^{-rT}N(d_2)$$

$$Rho_p = KTe^{-rT}N(-d_2)$$

This is not going to matter except in the case of extreme interest rate moves. I usually assume 0.01 for *r* and forget about it.

Mudd Finance

The BS/DE and what is says about the importance of the Greeks in determining an options price (when *r* = 0):

Theta

$$\Theta + P_0\Delta + \frac{1}{2}\sigma^2 P_0^2\Gamma = 0$$

The role played by: **Time** **Delta** **Stock Price** **Volatility** **Gamma**

When *r* is positive: see Hull 17.4.

$$\Theta + rP_0\Delta + \frac{1}{2}\sigma^2 P_0^2\Gamma = r\Pi$$

Mudd Finance

But then why bother .... ?

Call Option Price Calculator (Daily Volatility)

Stock symbol:	IWM
Call option:	May
Date Today:	4/19/2011
Expiration Date:	5/21/2011
DTM:	32
Stock Price:	82.10
Strike Price:	83.00
Daily Volatility:	0.0088
Interest Rate:	0.010
Time:	32
d1 Numerator:	-0.01002
Duration Volatility:	0.04978
Delta N(d1):	0.4202
N(d2):	0.4009
Option Price:	1.26
Option Premium:	1.26

Once you have any kind of option or derivative pricing model set up with all of the key variables represented, then it is trivial to tweak with loops such that it can use sensitivity analysis to measure the response to

- any variable with small ***or large*** changes
- any complex of variables simultaneously

Memo slides (not covered in lecture, not on exam)

Mudd Finance

Derivation of B/S DE using Ito's Lemma

and the relation to the Greeks (next five slides)**:**

Remember Ito's Lemma:

$$dG = \left(\frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S \, dz$$

Now lets define G to be *c*, the price of an option sensitive to S and change the formula to its discreet version:

$$\Delta c = \left(\frac{\partial c}{\partial S} \mu S + \frac{\partial c}{\partial t} + \frac{1}{2} \frac{\partial^2 c}{\partial S^2} \sigma^2 S^2 \right) \Delta t + \frac{\partial c}{\partial S} \sigma S \, \Delta z$$

Now we are going to construct a perfectly hedged portfolio using this derivative so that our volatility term will vanish.

Mudd Finance

Memo slides: See section 13.6 of Hull for explanation.

The Risk-free Ito Portfolio

Define the ***delta*** of an option to be the sensitivity of the option price to the price of the underlying stock and the ***gamma*** to be the rate of sensitivity:

$$delta = \partial c / \partial S \qquad gamma = \partial^2 c / \partial S^2$$

For example, if *delta* is 0.80, if the stock price rises by $1, the options price rises by 80 cents.

For example, delta is 2, then for each two shares of stock you short one option. If the option is $2 and the stock $10, then this is $18.

A risk-free portfolio exists if you short one derivatives contract (forget the boundary conditions) and offset it with a long share holding equal the delta. By definition, the value of your portfolio is equal to

$$\Pi = -c + \partial c / \partial S \; S \qquad \Delta \Pi = -\Delta c + \partial c / \partial S \; \Delta S$$

Mudd Finance

Some algebra

Substituting $\Delta S = \mu S \Delta t + \sigma S \Delta z$

and $$\Delta c = \left(\frac{\partial c}{\partial S} \mu S + \frac{\partial c}{\partial t} + \frac{1}{2} \frac{\partial^2 c}{\partial S^2} \sigma^2 S^2 \right) \Delta t + \cancel{\frac{\partial c}{\partial S} \sigma S \Delta z}$$

into $\Delta \Pi = -\Delta c + \partial f / \partial S \; \Delta S$

gives us $$\boxed{\Delta \Pi = \left(-\frac{\partial c}{\partial t} - \frac{1}{2} \frac{\partial^2 c}{\partial S^2} \sigma^2 S^2 \right) \Delta t}$$

Mudd Finance

... and more

The risk-less portfolio must return the same level of return as a portfolio earning the risk-free interest rate, therefore

$$\boxed{\Delta\Pi = r\Pi\,\Delta t}$$

Substituting these two equations into the new equation above

$$\Delta\Pi = \left(-\frac{\partial c}{\partial t} - \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2\right)\Delta t$$

$$\Pi = -c + {\partial c}/{\partial S}\,S$$

gives us

$$\boxed{\left(\frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2\right)\Delta t = r\left(c - \frac{\partial c}{\partial S}S\right)\Delta t}$$

Mudd Finance

which rearranged is Black-Scholes Nobel-prize winning differential equation:

$$\boxed{\underbrace{\frac{\partial c}{\partial t}}_{\text{time decay}} + rS\underbrace{\frac{\partial c}{\partial S}}_{\text{delta}} + \frac{1}{2}\sigma^2 S^2 \underbrace{\frac{\partial^2 c}{\partial S^2}}_{\text{gamma}} = rc}$$