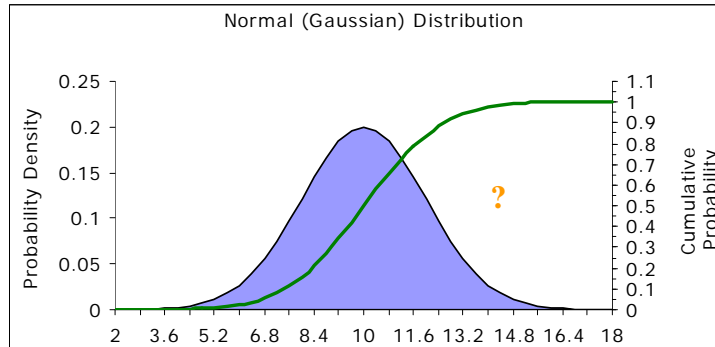


**Note:** This slide set was modified on April 26, 2011, to correct a mistake in the slide explaining the options pricing model.

**Mudd Finance**



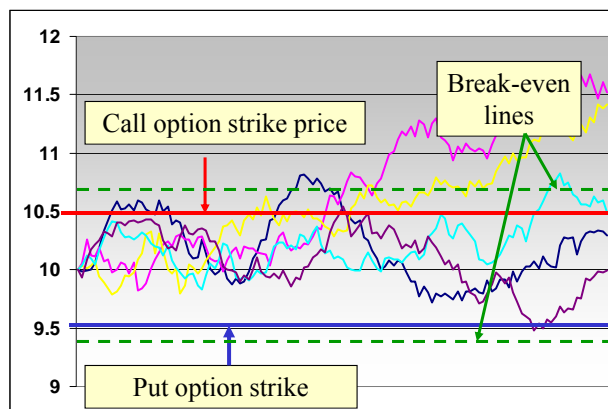
## The Black-Scholes Model

### ... pricing options and calculating Greeks

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## Monte Carlo Simulation of a Strangle



In this simulation suppose the stock is trading at 10 and we want to do a 6-month strangle at strike prices of 10.5 (call) and 9.5 (put). The stock has to go above or below these strike prices but we also have to cover our option costs (green line).

Again, this simulation does not include a Poisson (or equivalent) distribution, but should. Here, though, we don't have to wait until expiration and normally wouldn't. If we did, two of these make money, one has value but we lose money, and two expire worthless. What clearly matters? Volatility.

## The IWM strangle taken on April 7, 2011

Context: On Thursday April 7, heading into the weekend, there was a threat of a government “shutdown” because the GOP and DEMs could not agree and a new budget, Brent oil was above \$120, California gasoline above \$4 and rising, Portugal asking for a bailout, Libya still in turmoil, reactors leaking in Japan, and AA due to report on earnings (benchmark) after Monday’s market close, and the “storm of the millennium about to sweep across the U.S. Perfect for a 7-sigma.

Actual position taken at 9:55 AM PDT, IWM at 85.02, strangle consisted of May 21 86 Call priced at \$1.91, 83 Put priced at \$1.76, intended span is 6 days so time decay estimated at 34 cents, about 10%

Next slides show that historical volatility estimates are (1) one year 0.01562, (2) 60-day 0.01161, (3) 30-day 0.01167.

Implied call volatility was 0.01 (slightly underpriced) and put volatility was 0.01210 (about right).

Strategy: 3+ sigma news event, plus a mild hedge.

At 9:55 PST, Th 4/7/11

### Strangle/Straddle Option Value Calculator

Name: Gary R. Evans

Date: 4/7/2011

Stock Name: Russell 2000      Stock Symbol: IWM

Call Symbol: May 86      Put Symbol: May 83

Expiration Date: 5/21/2011      DTM: 44

Stock Price: 85.330      Days Time: 44

Call Daily Volatility: 0.01000      Put Daily Volatility: 0.01210

Interest rate: 0.010

CALL	PUT
Strike Price: 86.00	Strike Price: 83.00
d1 Numerator: -0.00661	d1 Numerator: 0.02890
Duration Volatility: 0.06633	Duration Volatility: 0.08026
N(d1): 0.46031	N(-d1): 0.35940
N(d2): 0.43409	N(-d2): 0.38981
Option Price: 1.99	Option Price: 1.65
Option Premium: 1.99	Option Premium: 1.65
Delta: 0.4603	Delta: -0.6406
Position Cost/Value: 3.64	Profit/Loss: 0.00
Original Cost: 3.64	Profit/Loss %: 0.00

Duration	
<b>One Year:</b>	
Average DGR:	0.00082
Standard Deviation:	0.01562
Average ABS DCGR:	0.01159
Min:	-0.05204
MaX:	0.05460
Min Norm DCGR:	-3.38376
Max Norm DCGR:	3.44253
<P9> to calculate	
<b>60 day:</b>	
Average DGR:	0.00130
Standard Deviation:	0.01161
Average ABS DCGR:	0.00893
Min:	-0.02604
MaX:	0.02294
Min Norm DCGR:	-2.35471
Max Norm DCGR:	1.86355
<b>30 Day:</b>	
Average DGR:	0.00225
Standard Deviation:	0.01167
Average ABS DCGR:	0.00912
Min:	-0.02604
MaX:	0.02294
Min Norm DCGR:	-2.42373
Max Norm DCGR:	1.77268

Strangle/Straddle Option Value Calculator			
Name:	Gary R. Evans		
Date:	4/7/2011		
Stock Name:	Russell 2000	Stock Symbol:	IWM
Call Symbol:	May 86	Put Symbol:	May 83
Expiration Date:	5/21/2011	DTM:	44
Stock Price:	85.330	Days Time:	38
Call Daily Volatility:	0.01000	Put Daily Volatility:	0.01210
Interest rate:	0.010		
	<b>CALL</b>		<b>PUT</b>
Strike Price:	86.00	Strike Price:	83.00
d1 Numerator:	-0.00677	d1 Numerator:	0.02873
Duration Volatility:	0.06164	Duration Volatility:	0.07459
N(d1):	0.45624	N(-d1):	0.35003
N(d2):	0.43190	N(-d2):	0.37804
Option Price:	1.83	Option Price:	1.48
Option Premium:	1.83	Option Premium:	1.48
Delta:	0.4562	Delta:	-0.6500
Position Cost/Value:	3.30	Profit/Loss:	-0.34
Original Cost:	3.64	Profit/Loss %:	-10.21

Using sensitivity analysis to calculate the 6-day time decay.

We plan to be in this before the deadline on the federal government budget shutdown and stay in it until, at the latest, after Alcoa earnings report at market close on Monday, April 7<sup>th</sup>.

As can be seen, the time decay will cost us \$0.34, or about 10%.

## The weekend strangle:



**PITraders** Gary R. Evans

Did IWM 85.02 6-day strangle May 21 86 C @ 1.91, 83 P @ 1.75, 6-day time decay 34 cents, C IDV 0.010, P IDV 0.0121, Hist 60 day DV 0.01161

7 Apr

Strangle/Straddle Option Value Calculator			
Name:	Gary R. Evans		
Date:	4/7/2011		
Stock Name:	Russell 2000	Stock Symbol:	IWM
Call Symbol:	May 86	Put Symbol:	May 83
Expiration Date:	5/21/2011	DTM:	44
Stock Price:	85.330	Days Time:	44
Call Daily Volatility:	0.01000	Put Daily Volatility:	0.01210
Interest rate:	0.010		
	<b>CALL</b>		<b>PUT</b>
Strike Price:	86.00	Strike Price:	83.00
d1 Numerator:	-0.00661	d1 Numerator:	0.02890
Duration Volatility:	0.06633	Duration Volatility:	0.08026
N(d1):	0.46031	N(-d1):	0.35940
N(d2):	0.43409	N(-d2):	0.38981
Option Price:	1.99	Option Price:	1.65
Option Premium:	1.99	Option Premium:	1.65
Delta:	0.4603	Delta:	-0.6406
Position Cost/Value:	3.64		
Original Cost:	3.64		

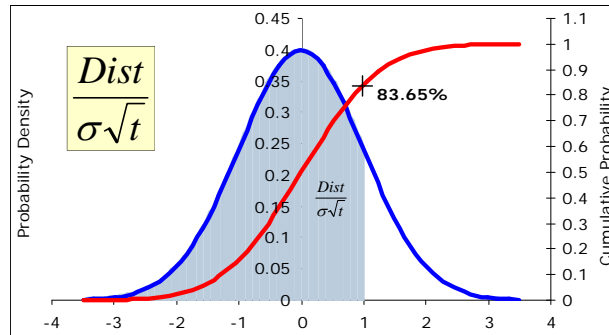


**PITraders** Gary R. Evans

Offset Friday IWM strangle, sold May 21 86 Call .89 (paid 1.91) and 83 Put 2.76 (paid 1.75) for net position loss of 1 cent.

4 hours ago

## The Standard Normal Distribution which will be useful



Dividing a normal distribution with mean 0 by its standard deviation produces the standard normal distribution, where we can describe the probability of a number being X standard deviations away from its mean. Shown is the probability of a value being less than +1 SD.

## Comparing the Normal and Lognormal distribution

1.  $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$  **Equation 1** shows the **normal probability density function (pdf)**. Assume that the mean is zero and the standard deviation is 0.10
2.  $g(x; \mu_x, \sigma_x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}$  **Equation 2** shows the **lognormal** pdf of  $x$ , where  $\mu$  and  $\sigma$  have the same value as in Equation 1. Compare 2 to 1.
3.  $Mean_g = e^{\mu+\sigma^2/2} = e^{0.005} = 1.005$  **Equation 3** is the mean and **equation 4** is the standard deviation of the lognormal distribution. Equivalent values are calculated.
4.  $SD_g = \sqrt{(e^{\sigma^2} - 1)e^{2\mu+\sigma^2}} = 0.100753$

Note the highlighted differences.

## About the relationship between the Normal and Lognormal Distributions

Thanks to Ben Preskill '08

If  $x$  is a random variable with a normal distribution, the  $y = e^x$  has a lognormal distribution. Likewise, if we consider a multiple of  $x$ ;

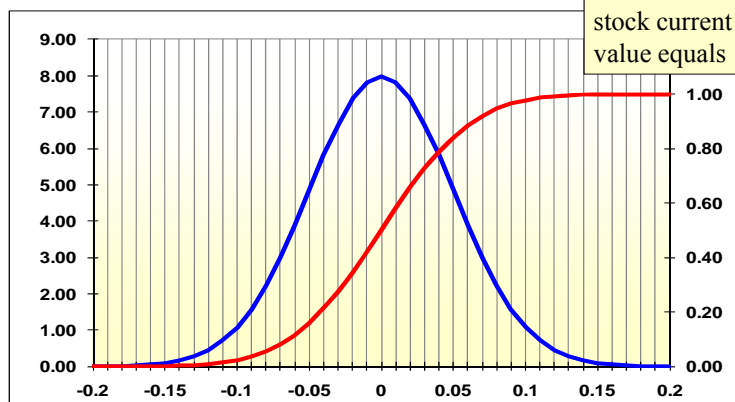
If  $x \approx N(\mu, \sigma^2)$  then  $\beta x \approx N(\beta\mu, \beta^2 \sigma^2)$

and  $\beta e^x \approx LNN\left(\beta e^{\mu + \sigma^2/2}, \beta^2 (e^{\sigma^2} - 1) e^{2\mu + \sigma^2}\right)$

So if in our case we assume  $r \approx N(0, 0.10)$  then this expression below

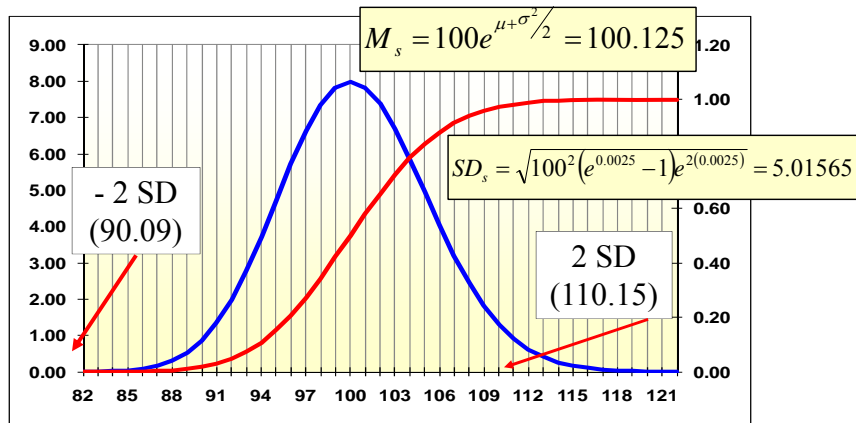
will be useful:  $S_i = S_0 e^r \approx LNN\left(S_0 e^{\sigma^2/2}, S_0^2 (e^{\sigma^2} - 1) e^{2\sigma^2}\right)$

## An assumed normal distribution for a stock continuous growth rate



This represents the assumed normal distribution for a the log continuous growth rate for a stock that has a sigma (volatility) of 0.05, between the ranges of +/- 4.0 sigma. Let's assume that this stock has a spot value of 100. What will happen if we translate these growth rates to stock prices?

## The resulting transformed log-normal distribution of stock prices



These are the same data, whereas the abscissa is transformed through the original distribution from growth rates to stock prices based and made symmetric. This distribution is log-normal. 100 is the *Median*.

## How Black-Scholes works ...

The Black-Scholes model is used to price European options (which assumes that they must be held to expiration) and related custom derivatives. It takes into account that you have the option of investing in an asset earning the risk-free interest rate.

It acknowledges that the option price is purely a function of the volatility of the stock's price (the higher the volatility the higher the premium on the option).

Black-Scholes treats a call option as a forward contract to deliver stock at a contractual price, which is, of course, the strike price.

In our treatment, let's consider **in-the-money call options of non-dividend paying stocks**.

## The Essence of the Black-Scholes Approach

- Only volatility matters, the  $\mu$  (drift) is not important.
- The option's premium will suffer from time decay as we approach expiration (Theta in the European model).
- The stock's underlying volatility contributes to the option's premium (Vega).
- The sensitivity of the option to a change in the stock's value (Delta) and the rate of that sensitivity (Gamma) is important [these variables are represented mathematically in the Black-Scholes DE, next lecture].
- Option values arise from arbitrage opportunities in a world where you have a risk-free choice.

## The Black-Scholes Model: European Options

$$C = SN(d_1) - Ke^{-r(t/365)}N(d_2)$$

C = theoretical call value  
 S = current stock price  
 N = cumulative standard  
     normal probability dist.  
 t = days until expiration  
 K = option strike price  
 r = risk free interest rate  
 $\sigma$  = daily stock volatility

$$d_1 = \frac{\ln(S/K) + (r/365 + \sigma^2/2)t}{\sigma\sqrt{t}}$$

$$d_2 = \frac{\ln(S/K) + (r/365 - \sigma^2/2)t}{\sigma\sqrt{t}}$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

**Note:** Hull's version (13.20) uses annual volatility. Note the difference.

### Breaking this down ...

$$C = SN(d_1) - Ke^{-r(t/365)}N(d_2)$$

This term discounts the price of the stock at which you will have the right to buy it (the strike price) back to its present value using the risk-free interest rate. Let's assume in the next slide that  $r = 0$ .

$$d_1 = \frac{\ln(S/K) + (r/365 + \sigma^2/2)t}{\sigma\sqrt{t}}$$

Dividing by this term (the standard deviation of stock's daily volatility adjusted for time) turns the distribution into a standard normal distribution with a standard deviation of 1.

### ... or simplifying it some

$$CP = SP \times \Pi(d_1) - STR \times \Pi(d_2)$$

This is the absolute log growth difference between the strike price and the stock price.

... assume that  $r$  is 0 and  $t$  is 1:

$$d_1 = \frac{\ln(SP/STR) + \sigma^2/2}{\sigma}$$

We are calculating the cumulative probability to this standard normal point.

This normalizes it to standard normal (the numerator is now "number of standard deviations.")

$\mu$  is zero so this is the log-normal zero mean adjustment

$$C = S \times N(d_1) - K \times N(d_2) \quad (\text{assuming } r \text{ to be } 0) \quad \text{Mudd Finance}$$

**... and some more**

This term, our  $x$  of two slides ago, represents the **spread in continuous growth terms** between the stock price and the strike price, and when normalized by the denominator, the spread as the number of standard deviations. For example, if  $S = 110$  and  $K = 100$  and volatility = 10%, then this term equals 9.5%, or about one standard deviation.  $x > 0$  for **itm calls** and **otm puts** and  $x < 0$  for **otm calls** and **itm puts**.

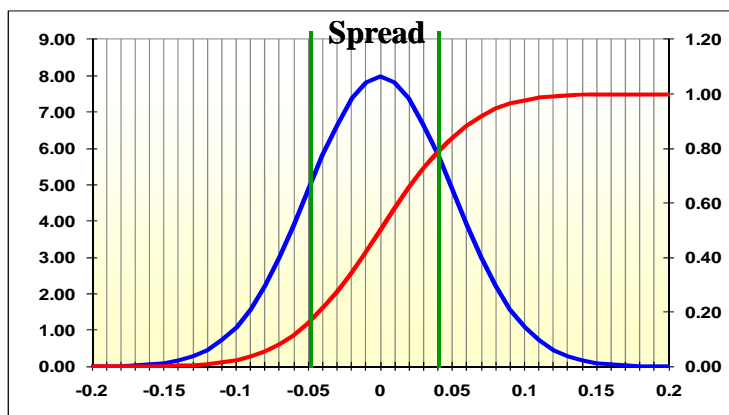
$$d_1 = \frac{\ln(S/K) + (r/365 + \sigma^2/2)t}{\sigma\sqrt{t}}$$

This term has the effect of removing the bias.

$$d_2 = \frac{\ln(S/K) + (r/365 - \sigma^2/2)t}{\sigma\sqrt{t}}$$

This term has the effect of centering the transformed lognormal distribution spread on the median price.

**What you are doing – centering the spread on the mean.**



You are calculating the probability in both  $d_1$  and  $d_2$  of the spread centered about the mean (adjusted to center on the mean).

## Calculating implied volatility with B/S:

$$d_1 = \frac{\ln\left(\frac{SP}{STR}\right) + \sigma^2/2}{\sigma}$$

$\sigma$

Very easy to do:  
Once Black-Scholes is structured, you can use an iterative technique to solve for  $\sigma$ .

### Call Option Price Calculator (Daily Volatility)

Stock symbol: INTC  
Call option: July  
Date Today: 5/11/2010  
Expiration Date: 7/16/2010  
DTM: 66

Stock Price: 23.97  
Strike Price: 26.00  
Daily Volatility: 0.0110  
Interest Rate: 0.010  
Time: 92

d1 Numerator: -0.07876  
Duration Volatility: 0.10551  
Delta N(d1): 0.2277  
N(d2): 0.1971

Option Price: 0.35  
Option Premium: 0.35

## Using the Black-Scholes Model

There are variations of the Black-Scholes model that prices for dividend payments (within the option period). See Hull section 13.12 to see how that is done (easy to understand). However, because of what is said below, you really can't use Black-Scholes to estimate values of options for dividend-paying American stocks

There is no easy estimator for American options prices, but as Hull points out in chapter 9 section 9.5, with the exception of exercising a call option just prior to an ex-dividend date, "it is never optimal to exercise an American call option on a non-dividend paying stock before the expiration date."

The Black-Scholes model can be used to estimate "*implied volatility*". To do this, however, given an actual option value, you have to iterate to find the volatility solution (see Hull's discussion of this in 13.12). This procedure is easy to program and not very time-consuming in even an Excel version of the model.

For those of you interest in another elegant implied volatility model, see Hull's discussion of the IVF model in 26.3. There you will see a role played by delta and vega, but again you would have to iterate to get the value of the sensitivity of the call to the strike price.

$$=LN(SP/KP)+(IR+(DV*DV)/2)*(DTM/365)$$

$$=LN(SP/KP)+((IR/365)+(DV*DV)/2)*DTM$$

### An example ...

Consider an itm option with 20 days to expiration. The strike price is 105 and the price of the stock is 100 and the stock has an daily volatility of 0.02. Assume an interest rate of 0.01 (1% annual).

$$d_1 = \frac{\ln(100/105) + (r/365 + 0.02^2/2)20}{0.02\sqrt{20}} = -0.49464$$

$$d_2 = d_1 - 0.02\sqrt{20} = -0.58409$$

$$C = 100N(-0.04424) - 105e^{-0.01(20/365)}N(-0.58409) = 1.70$$

### Using an Option Value Calculator to Calculate this same Value

Call Option Price Calculator (Daily Volatility)	
Stock symbol:	Trial
Call option:	May
Date Today:	4/26/2011
Expiration Date:	5/16/2011
DTM:	20
Stock Price:	100.00
Strike Price:	105.00
Daily Volatility:	0.0200
Interest Rate:	0.010
Time:	20
d1 Numerator:	-0.04424
Duration Volatility:	0.08944
Delta N(d1):	0.3104
N(d2):	0.2796
Option Price:	1.70
Option Premium:	1.70

NUM

$$=LN(SP/KP)+((IR/365)+(DV*DV)/2)*DTM$$

DUV

$$=NORMSDIST(NUM/DUV)$$