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Is Uncle Norm's shot going to exhibit a Weiner Process? Knowing Uncle Norm, probably, with a random drift and huge volatility.

Dynamic Modeling

Weiner Processes and Ito's Lemma

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About the reading

Reading about this material is essential. The book will add a lot of explanatory power to my lectures.

We are not going to cover the material on binomial models.

Immediately read chapter 12 (Weiner Processes and Ito's Lemma), and next we will cover chapter 13 (Black-Scholes). I am trying to stress *applied modeling* at this point.

Also: Go to http://www.cboe.com and find their options calculator (easiest to find with "CBOE options calculator" in search). If flaky Java is not cooperating, find another. Take a look at it and prepare to build one after the Chapter 13 lecture.

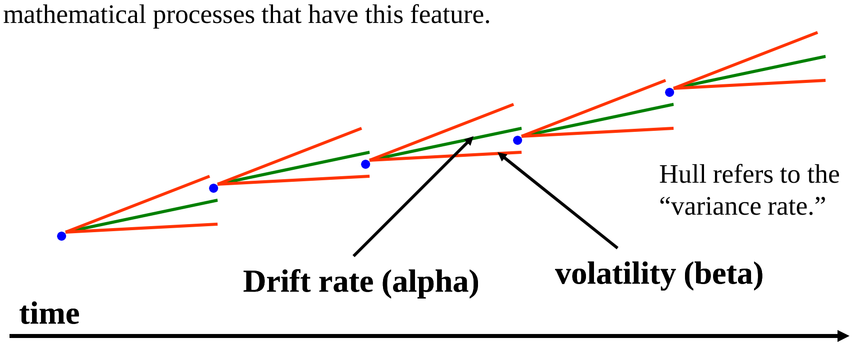
Setting up

Remember that we are regarding the time-series stream of financial data that we are processing as a Markov process, which means that we regard the process as continuous, any data that we might be using as a sample from a continuous population, and far more important, each observation at times "t" is completely independent (in the mathematical sense) of all prior observations except the immediately prior observation (the latter "immediately prior" condition is necessary because otherwise we wouldn't have a process, we would have total chaos).

We also need to remember that when we add two ***independent*** random variables that have a normal distribution, then the resulting mean is the sum of the means and the resulting variance is the sum of the variances, which, given that variance is always $0<V<1$, the resulting standard deviation, which is the square root of the summed variances, will be less than the sum of the two standard deviations.

About drift and volatility (again – last time)

We are going to regard the path of securities (and their derivatives) prices as a Markov Process with actual price behavior over time reflecting ***drift*** and ***volatility***, where the latter is represented by a Gaussian distribution. The resulting pattern will reflect randomness with a trend. We are searching for mathematical processes that have this feature.



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Weiner Processes

A Weiner Process is fairly elementary. It is used in physics to model *Brownian Motion*, which is the motion of an itty bitty particle when it is impacted by some kind of shock. A Weiner Process is a special type of Markov Process with a mean change of zero (no drift) and a *variance and volatility rate* of 1.0 per year distributed in a standard normal probability distribution N(0,1). Any variable z follows a Weiner process if it is a Markov process and:

The change in z over a small period of time is $\Delta z = \varepsilon\sqrt{t} = \sqrt{t}\ N(0,1)$

Large-scale properties (definitional):

$$\mu_{z(T)} = 0$$

$$\sigma^2_{z(T)} = T$$

$$\sigma_{z(T)} = \sqrt{T}$$

Refers to $z(T) - z(0)$ over a long time like one year. Note that when T=1 that both the variance and standard deviation are 1.

Small-scale Properties:

$$\mu_{\Delta z} = 0$$

$$\sigma^2_{\Delta z} = \Delta t$$

$$\sigma_{\Delta z} = \sqrt{\Delta t}$$

One day would be 1/252.

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Generalized Weiner Process

A generalized Weiner Process is represented by the equation

$$ds = a\ dt + b\ dz$$

Because a Weiner process has a variance and standard deviation of 1.0 annually, then the expression above has a standard deviation of b. In a small time interval

$$\Delta x = a\Delta t + b\varepsilon\sqrt{\Delta t}$$

$$\mu_{\Delta x} = a\Delta t \qquad \sigma^2_{\Delta x} = b^2\Delta t \qquad \sigma_{\Delta x} = b\sqrt{\Delta t}$$

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The Ito Process

… is a useful extension of the Weiner Process and is written as

$$dx = a(x,t)\,dt + b(x,t)\,dz$$

where

$$\Delta x = a(x,t)\Delta t + b(x,t)\varepsilon\sqrt{\Delta t}$$

The drift and variance rate are *a* and b^2,volatility is *b*.

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Discrete Ito Process* for Stock Price Behavior

$$\frac{\Delta S}{S} = \mu_s \Delta t + \sigma_s \varepsilon \sqrt{\Delta t}$$

$$\Delta S = \mu_s S \Delta t + \sigma_s S \varepsilon \sqrt{\Delta t}$$

(see 12.7, 12.8, and 12.9 and example 12.3in the book).

Symbol	Example	(*geometric mean and SD*)
S	10	: a scalar (stock price)
μ	1.08	: mean return or drift *for one year.*
σ	0.12	: volatility (standard deviation) *for one year.*
T	1	: one year, or 250 days.
Δt	0.004	: short time, typically one day, or 1/250.
$\sqrt{\Delta t}$	0.0632	: square root of delta t.
ε	$\Phi(0,1)$	: random selection from standard normal probability distribution.

Look at what these two equations are saying, especially the right one. You can plainly see the drift and volatility and the adjustment for time. Also, clearly, because of its derivation and the constant volatility assumption, any application of this is relevant only for short periods of time.

*also referred to as geometric Brownian series.

A personal note about the equation on the previous page

Because I don't want to confuse you (too much) the equation on the previous page is presented that way because that is exactly how Hull presents it, assuming that the volatility estimate is for a year. Since I use daily volatility, which might be daily classical historical volatility or even daily variable dynamic volatility and I simulate day by day (which implies that $\Delta t = 1$), my version of the formula for simulations looks like:

$$S_{t+1} = S_t + \Delta S$$

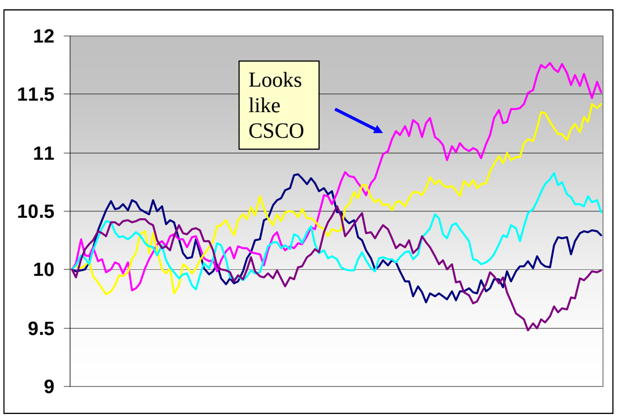
$$\Delta S = \mu_{ds} S_t + \sigma_{ds} S_t \varepsilon$$

If using annual volatility as Hull does, then:

$$\sigma_{ds} = \sigma_{as} \big/ \sqrt{252}$$

As we will soon see, though, there is a bias in our drift estimator, so we need to tweak this equation a little more.

Monte Carlo Simulation of Ito Process



This six-month simulation was done in Excel assuming that the starting stock price was 10, the stock had an annual drift of 8%, and annual volatility of 12%, the year has 250 days, and each delta t is one day, or 0.004. This is from equation 12.8 in the book.

This is a five-time simulation of the discrete-time Ito Process with parameters shown on the previous page (see Hull page 271 for procedure). The random sample from the standard normal distribution was generated by the command NORMINV(RAND()). This simulation looks a little biased to the high side, but five others might look a little biased to the low side.

*derivation is shown in Appendix to chapter 12.

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Ito's Lemma

Assume that the value of a variable *x* follows the Ito process

$$dx = a(x,t)dt + b(x,t)dz$$

where *dz* is a Weiner process. Then Ito's Lemma* (1951) states that any function G of *x* and *t* follows this process below:

$$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b \, dz$$

Substituting μS and σS for *a* and *b* and *S* for *x:*

$$dG = \left(\frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S \, dz$$

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Applying Ito's Lemma to Ln(S)

$G = \ln S$	$\frac{\partial G}{\partial S} = \frac{1}{S} = S^{-1}$	$\frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}$	$\frac{\partial G}{\partial t} = 0$
starting assumption	log function rule	from equation to left	G not defined in t

Substituting the expressions above into

$$dG = \left(\frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S \, dz$$

gives us

$$dG = \left(\frac{1}{S} \mu S + 0 - \frac{1}{2} \frac{1}{S^2} \sigma^2 S^2 \right) dt + \frac{1}{S} \sigma S \, dz$$

reducible to

$$dG = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma \, dz$$

and we still have our drift and volatility.

... about the final term

Because we are assuming that $\Delta \ln S$ is normally distributed through time and the appropriate "one day" variance is best represented as σ^2, we can replace the final dZ term with the one shown below, resulting in the final relevant form of our equation:

$$dG = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma\varepsilon$$

where ε is a sample from a standard normal distribution in a simulation.

So where are we?

Remember that we converted prices to their natural logs or continuous growth rates using natural logs. In either case we assume that the resulting distribution is normal.

Our tweaking of the Ito Process and the application of Ito's Lemma suggests to us that when we take the delta of the natural log of a stock price or the difference in logs (the CGR), the following terms for drift and volatility emerge:

$$\left(\mu - \frac{\sigma^2}{2} \right) \qquad \sigma\varepsilon$$

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The Expected Value of our Bet

And on top of this (previous slide) interesting issue, Hull introduces us to the concept of the ***Expected Value*** of a stock bet (sec 13.3), a concept you should be familiar with (what is the expected value of rolling a pair of dice)? If

$S_f = S_0 e^{(\mu+\sigma\varepsilon)T}$ then $E(S_t) = S_0 e^{\mu T}$ because $E(\sigma\varepsilon) = 0$

Well, doesn't this imply that the expected growth rate of *S* is therefore μ ?

No. (See Hull's example by logs at bottom of page 281, which is different than this). Whereas $\sigma\varepsilon$ has a normal distribution by definition, so therefore its expected value is zero, *S* has a log-normal distribution and the volatility of S is not distributed across a normal distribution.

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The Unbiased Ito Monte Carlo Simulation

If we are using daily volatility and the daily mean growth rate (or some alpha estimator for the mean growth rate, or even zero for the mean growth rate), then the unbiased estimator for the simulation will be:

$S_{t+1} = S_t + \Delta S$

$$\Delta S = \left(\mu_{ds} - \sigma^2/2\right)S_t + \sigma_{ds} S_t \varepsilon$$

implies that we are simulating from a standard normal distribution, N(0,1) if we use Excel then we can take a sample from that distribution using Excel command NORMSINV(RAND(),MEAN,STDV). The internal random term takes a random sample from the range of 0 to 1. The combined term takes a sample from the normal distribution *weighted by the probability.* (Note: this does not mean that the sample will be within one standard deviation of the mean - it will be spread throughout the distribution).

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But what about those six sigma events??

As we know, most data movements that are described as six sigma events (or equivalent) just because they represent a move six standard deviations from a historical estimator, are not six sigma events in any meaningful way. They happen because

1. Volatility is a variable, and can become a volatile variable,
2. Processes that were though to be statistically independent turn out not to be,
3. Stock prices and markets actually respond to real events, including shocking or surprising events, not to historical data.

Was Sunpower's 20% (10 sigma) plunge on November 6 beyond explanation? Was it really improbable? No. It was due to the severe Euro devaluation of that month ... they do most of their business in Europe and the devaluation hammered their profits.

So is there any way to model this??

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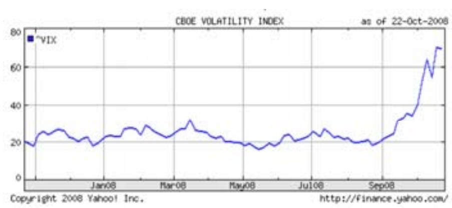
Issue 1: Volatility is a variable

The tracking model described so far assumes that volatility, no matter how estimated, is a constant over the duration of the simulation.

But in October 2008, which saw daily "six sigma events" we know that volatility for the most staid stocks and even diversified indexes can be highly variable.

The VIX chart in the upper right shows that.

There are various ways to consider variable volatility:



Volatility is a function of economic variables.

$$\sigma = f\left(\bar{x}\right)$$

Volatility is proportionate (or a function of) the normalized VIX multiplier.

$$\sigma_a = \alpha VM$$

Volatility has its own normal distribution, which can be sampled in a simulation.

$$\sigma \approx N\left(0, \sigma_v^2\right)$$

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Issue 2: Assumptions of Independence are False

$$VAR\left[\sum_{i=1}^{n} \alpha_i x_i\right] = \sum_{i=1}^{n} \alpha_i^2 VAR(x_i) + 2\sum_{i=1}^{n-1} \sum_{j=(i-1)}^{n} \alpha_i \alpha_j COV(x_i, x_j)$$

This danger is more common to portfolios than individual stocks, but financial "contagion" can impact almost any kind of financial asset, including those with a history of independence and stability.

The huge financial failures, such as the LTC hedge fund failure and the market contamination caused by the CMO crisis in 2007/2008 are prime examples of this.

These ***were not*** "six sigma events." They were caused by the breakdown of assumed independence.

The equation above shows the variance of a portfolio that has maxed the Sharpe Ratio by finding small values for the CORREL in the COV term.

During contagion, the CORREL swells up and the volatility explodes.

This happened in October 2008.

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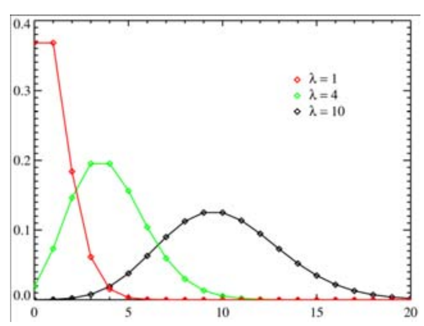
Issue 3: Adding a Poisson Distribution

$$P(k;\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Lamda is the expected number of events in this time interval.
K is the number of events in this interval.
P is the probability of k events given lamba during this time interval.

This allows for a mapping of a discrete distribution and a cumulative distribution function.



Source: wikipedia.com

This is likely to be useful if modeling the time-path of a stock on which you are doing an options trade like straddles, strangles, or writing covered calls. The model shown above is a homogeneous model where lamba is assumed to be constant. Sometimes a more applicable model would be a non-homogeneous model where lamda is a function of time (like right after an earnings report). Teacher’s comments.

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Notes on these "events"

In the previous slide ***k*** is a non-valued event, and it is binary, it either happens or it doesn't happen. So the probability estimator tells us the probability that ***k*** will happen 3 times or 6 times or no times over the time interval in question (like to 12 days up until options expiration). Fortunately, if ***k*** needs to only happen once for, say, a straddle to have value a covered call to be exercised, all we care about is the probability that ***k*** equals zero, where the probability of our trigger event is one minus that probability. For example, if the probability that ***k = 0*** is 0.40, then the probability that something will happen to make our straddle profitable or put our written call in the money will be 0.60 (I am simplifying here - it might also happen because of volatility).

But we have to give the event value. If we are saying that the likelihood of at least one spike is 0.60, by how much will that spike move the stock price on average? Generally, as much as it has moved it on average in the past?

And how is the ***k*** event defined? Maybe any movement of 3+ standard deviations? (Teacher's comments)

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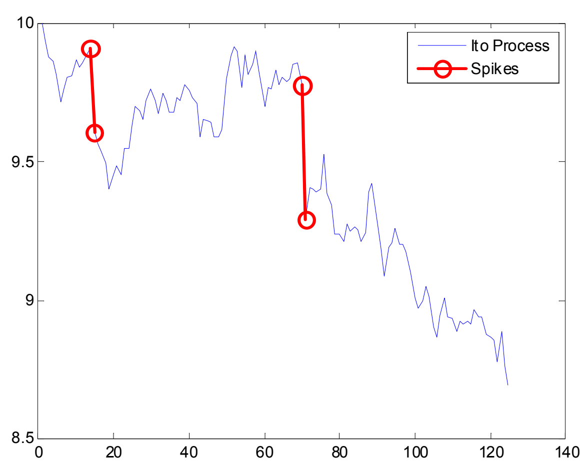
Student contributions

Contributed by Philip Amberg, December 2007, written in C++.



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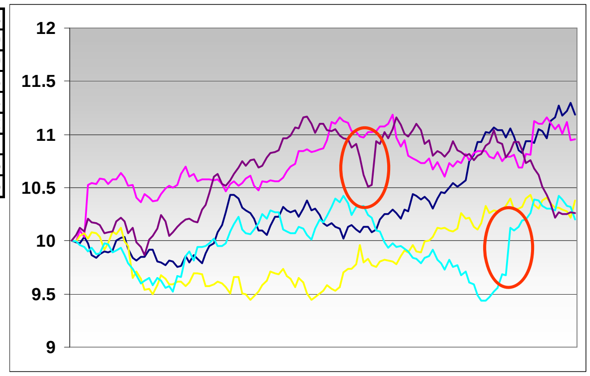
Student contributions (cont)

Jonathon Litz, 2008, simple Werner/Ito process

Annual Mean Return	1.08
Annual Volatility	0.10
One day	0.004000
SQRT One day	0.063246
Lambda	1.4
Lamdba / Day	0.0112
k	1
Prob of 1 Event on 1 Day	0.011075
Number Years	0.5



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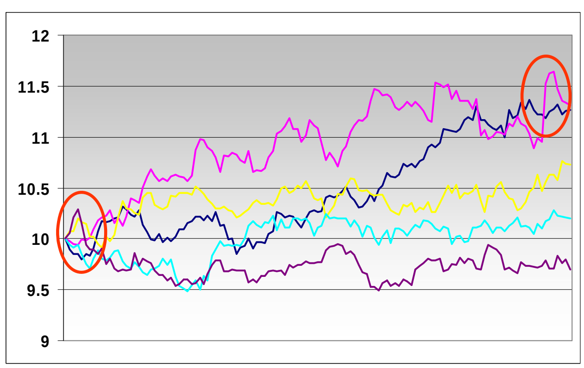
Student contributions (cont)

Greg Minton, 2008, jump size randomly determined

Annual Mean Return	1.08
Annual Volatility	0.10
One day	0.004000
SQRT One day	0.063246
λ for whole period	1.4
Number of days	125
Prob on a given day	0.0112
Mean jump size	3.75
Stdev jump size	1.25

Note: Random jump size



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Portfolio Volatility and Monte Carlo Diversification Simulations

The slides that follow demonstrate the benefits of diversification using Vanguard's S&P500 Index fund **VFINX** and Vanguard's Intermediate Term U.S. Treasury Bond Fund, **VFITX**.

We take advantage of the sum of weighted variances:

$$V(aX + bY) = a^2V(X) + b^2V(Y) + 2abCOV(X,Y)$$

Remembering that statistically covariance is defined to be equal to the correlation coefficient of X and Y times the product of their standard deviations:

$$COV(X,Y) = COR(X,Y)SD(X)SD(Y)$$

we will achieve diversification only if X and Y are largely ***independent***!

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How we do this …

1. Download five years of weekly historical data for VFINX and VFITX.
2. Take natural logs of the data and calculate weekly historical growth rates.
3. Calculate the weekly geometric mean growth rates for both.
4. Calculate the weekly variance and standard deviations of those growth rates for both.
5. Simulate your portfolio behavior with Monte Carlo simulations (of a Werner process) using weights that sum to one.

The formulas on the previous page produce the result that so long as your two investments are independent in their behavior, the volatility of your portfolio will be far less than the volatility of the most volatile component of your portfolio, and perhaps less volatile than the least volatile component of your portfolio!

For example, if you had two independent assets with the identical growth rate of 0.06 and standard deviations of 0.02, if you invested have of your money in each, the portfolio would still have a growth rate of 0.06 but a standard deviation of 0.1414.

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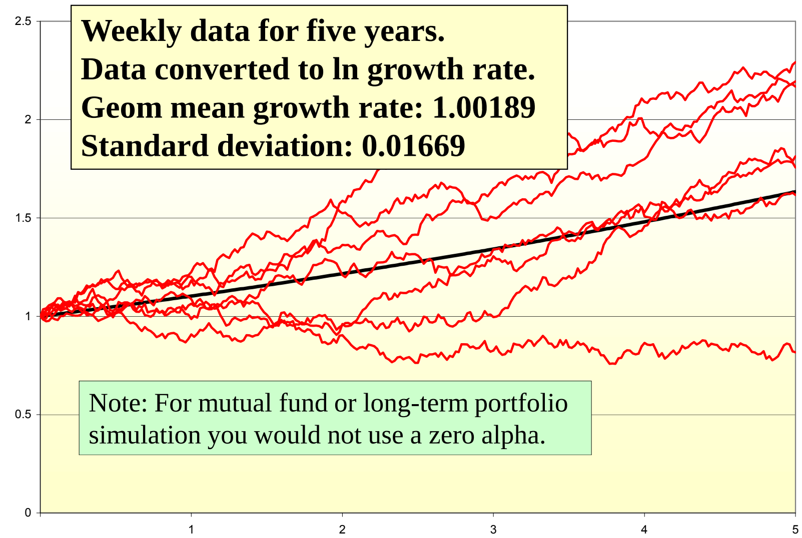
VFINX (S&P 500) 5 year MC



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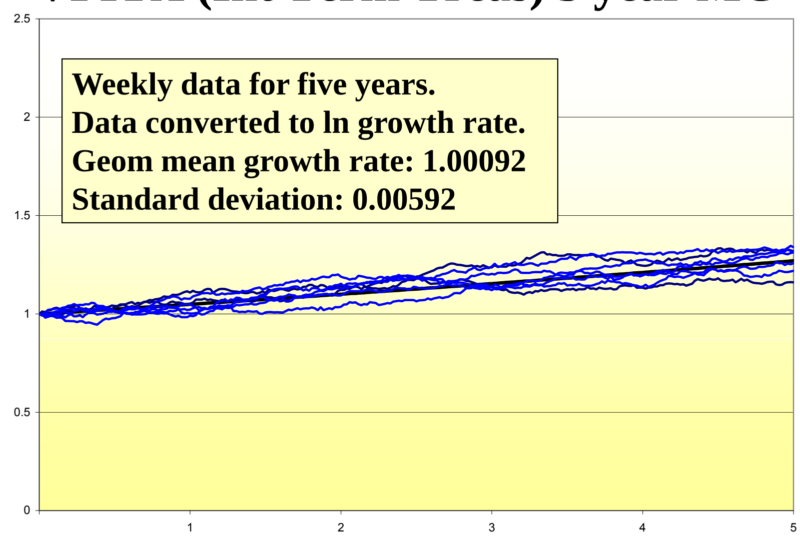
VFITX (Int Term Treas) 5 year MC



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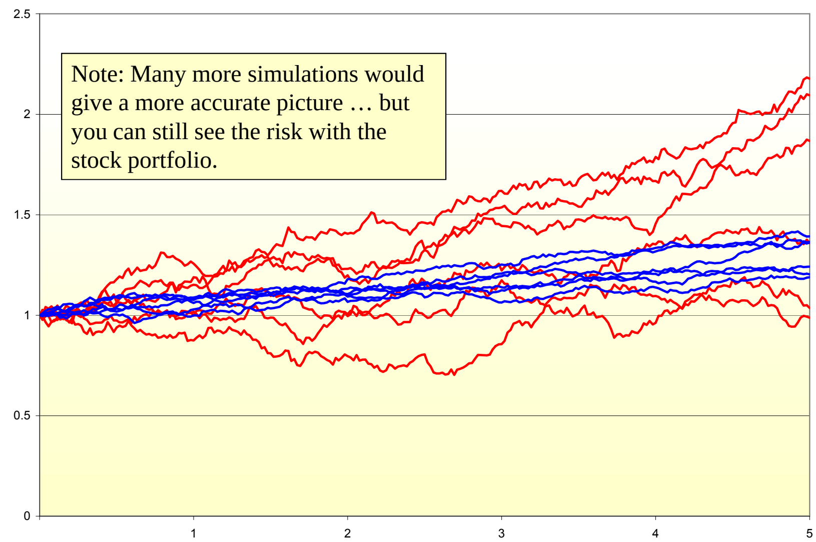
VFITX & VFINX Together



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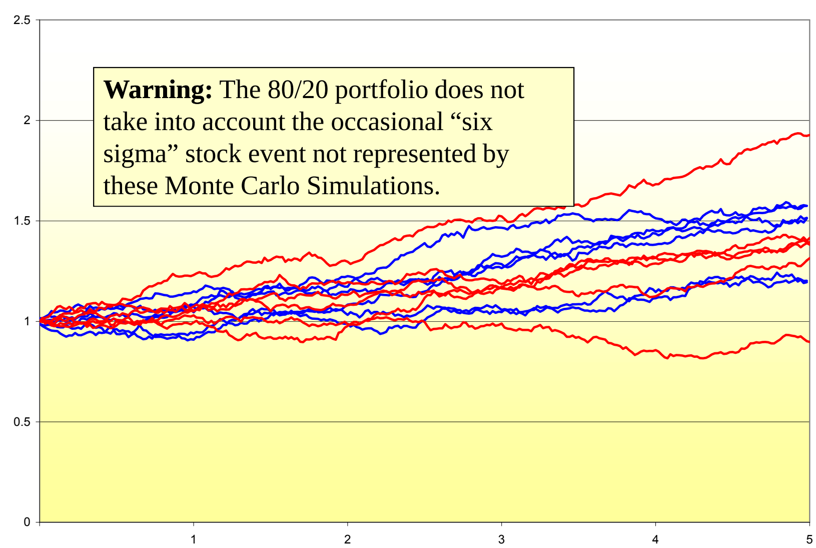
An 80/20 vs. 50/50 Portfolio



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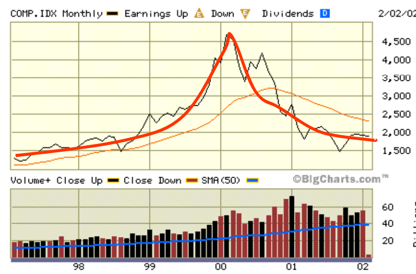
Building a Momentum Ramp

Here is a good project for an ambitious student: How would you build a momentum ramp into a simulation model?

Momentum happens when an *alpha* begins to steadily rise, often for a legitimate reason. Because of the gains made, especially on leveraged trades, momentum takes over.

This has happened to many individual stocks (GOOG), to the stock markets twice in the last decade, oil and commodities in 2008, and real estate.



$$\mu_t = f\left(\mu_{lagged}\right)$$

The alpha (our *mu*) would be a lagged function of itself, but a threshold variable *normally stable* that triggers off past the threshold. It would also have to include a *break point* built in.

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The much-altered simulator

$$\Delta S = \left(\mu_{ds} - \sigma^2/2\right) S_t + \sigma_{ds} S_t \varepsilon$$

+

$$\sigma \approx N\left(0, \sigma_v^2\right)$$

Make variance variable either this way or as sensitive to the VIX.

CORREL

Especially if this is a portfolio value, make variance a function of correlations, then jump the correlations (not easy).

$$\Omega\left(\bar{x}\right)$$

Stocks rise and fall in response to economic variables. A real model would attempt to simulate that and might also have momentum built in (lagged alpha structure).

$$P(k;\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Put a Poisson (or similar) boolean tail on the simulator, ideally with a random jump size.

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Do simulators have any predictive power?

Certainly not much, but maybe some. If you have **adjusters** in the model for **jumps**, **momentum**, **contagion** and **sensitivity to general market volatility** (VIX) that is based upon some kind of reasonable empirical review, then at least the simulation will show you the possibilities that emerge.

After reviewing such simulations, maybe you won't make the stupid decision of undertaking high-risk leveraged portfolio bets blithely assuming that the historical statistical independence between your portfolio and other market assets will maintain through a period of crisis. Maybe a good simulation will instruct you to hedge or moderate your bets, or at least look for clear signs of danger that correlations are changing.

These models do have strong pedagogical possibilities for such things. And just like looking at the chart for a stock over a day's or year's trading, even if you are not a "market technician," you can see some instructive and revealing patterns. Maybe they can give you an edge.

The real trouble with any model: they can't pick turning points.