Base and Subbase in a Number System

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Introduction

When people say that we in the United States use a base ten number system, there are two senses in which that statement is to be interpreted. First, linguistically, when we use English, the structure of the number words shows a base of ten, at least at the outset. Second, when we write numbers in our version of Arabic numerals, the structure of our number symbols also shows a base of ten. I will explore the notion of the base of a number system (verbal or numeral) in this note, and also introduce the auxiliary notion of subbase.

Systems of Number Words

In English we have distinct number words for the numbers one through ten. The words eleven and twelve are exceptions, but then thirteen through nineteen are formed using a base related to the words for the units and a suffix meaning add ten. Multiples of ten from twenty through ninety are formed with a base corresponding to the multiplier and a suffix meaning multiply by ten; units are added on as a second part of the word. Thus we have thirty-four (for three tens plus four) and seventy-six (seven tens plus six), for example. A new word, hundred, is introduced for ten tens, or 102, but then multiplicative and additive principles enter in again. The next totally new word is a thousand, for ten hundreds or 103, and then again multiplicative and additive principles appear. The fact that new words are introduced for powers of ten and that intermediate words are formed using multiplicative and additive principles leads one to refer to the system of number words as a base ten system.

If we look at French number words we also see a base ten structure, but with more exceptions. The table below shows some French number words up to one thousand.

French Number Words

1	un	10	dix	20	vingt
2	deux	11	onze	30	trente
3	trois	12	douze	40	quarante
4	quatre	13	treize	50	cinquante
5	cinq	14	quatorze	60	soixante
6	six	15	quinze	70	soixante-dix
7	sept	16	seize	80	quatre-vingts
8	huit	17	dix-sept	90	quatre- vingts-dix
9	neuf	18	dix-huit	100	cent
		19	dix-neuf	1000	mille

One could argue that number words for eleven through sixteen show a base for the units and a suffix meaning add ten, but the construction is very different for seventeen, eighteen, and nineteen. The completely new word for twenty does not follow a base ten pattern, which would be a construction displaying two tens (does it show base twenty instead?). The words for eighty and ninety would support a base twenty interpretation, but this reading of the words is denied by the form of the words for thirty through sixty, which do show a base ten form. The word for seventy is exceptional in any case. There are new words introduced for one hundred and one thousand, but otherwise, for hundreds and thousands number words are formed using multiplicative and additive principles.Thus one gets trois cents soixante-deux for 362, for example. This use of multiplicative and additve principles reinforces the base ten interpretation, and we conclude that French number words show a base ten structure with exceptions, some of which may point to remnants of a base twenty structure.

As a test of our ability to deduce the base of a system of number words let us consider the number vocabulary of the Nengone culture of the Loyalty Islands in Melanesia. A table of some of their number words follows (see Codrington).

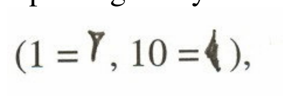
Table of Nengone Number Words (Loyalty Islands, Melanesia)

1	sa	10	rewe tube nine
2	rewe	11	rewe tube ne sa
3	tini	20	re ngome
4	ece	21	re ngome ne sa
5	se dongo	30	sa re ngome ne rewe tube nine
6	dongo ne sa	40	rewe re ngome
7	don go ne rewe	100	se don go re ngome
8	don go ne tini		
9	dongo ne ece		

Words for the numbers from one through five are distinct. Numbers six through nine are obviously combinations of the word for five and the extra units being added to five. The expression for ten is two of something, not the former word for five but probably an alternate. The system continues to look like a base five system up to twenty, when an unexpected and new word is introduced, clearly neither four fives nor two tens. Beyond twenty multiples of ten and twenty show a base twenty structure. We might conclude that up to twenty this system is a base five system, but from twenty on it shows a base twenty structure. Such a system is said to have base twenty and subbase five.

Base in Numeral Systems

The second way in which people speak of our number system as a base ten system refers to our symbolic way of representing numbers, our numerals. In this sense we have distinct numerals for numbers zero through nine, and shift digits positionally to indicate multiples of different powers of ten. Thus 1,507 represents one thousand five hundred seven. To further illustrate the notions of base and subbase in a numeral system I want to examine the Babylonian numeral system, the symbols used to represent numbers in Mesopotamia around 4,000 years ago. Writing in this culture was accomplished by pressing a stylus into a soft clay tablet; to preserve the work, the tablet was baked when completed. Two different stylus imprints gave symbols for one and ten



and these symbols were repeated an appropriate number of times to get numbers up through fifty-nine. In forming these numbers, symbols for ten were placed to the left in a cluster, symbols for one were frequently stacked up in up to three rows on the right, and each symbol would be touching one or more of theother symbols. A couple of examples follow.

Thus up to 60 this numeral system looks like a base ten system--there are different symbols for one and ten, and an additive principle works to get intermediate numbers. But the system is in fact a positional base sixty system, not a base ten system. Numbers are expressed in terms of sums of multiples of powers of sixty, and represented with the numerals for one to fifty-nine in appropriate positions, higher powers of sixty to the left. Thus four hundred thirty-two would be thought of as seven sixties plus twelve, and would be writen with the numerals for seven in the sixties position (to the left), followed by the numerals for twelve in the units position (to the right):

As another example,

 $= 3 \times 60^2 + 32 \times 60 + 41 = 12{,}761.$

Using the terminology of base and subbase, Babylonian numerals form a base sixty system with a subbase of ten--seemingly base ten up to sixty, where the true base becomes evident. Otto Neugebauer, an early translator of and commentator on Babylonian science texts in the last century, thought it desirable to keep base sixty representations of numbers but to make them easier to write and comprehend than they are in Babylonian numerals. Thus he adopted the notation of representing the numbers from one to fifty-nine (actually zero to fifty-nine) using our familiar numerals, and separating multipliers of successive powers of sixty by commas. Thus the number above, 32 * 60^2 + 32 * 60 + 41, is written 3,32,41 in this notation (see Neugebauer).

Common Features

At this point I should provide definitions for the notions of base and subbase of a number system, and do it with sufficient generality to cover both the case of a system of number words and a numeral system. I would probably have more success if I assume, since I have been using the terms, that you know the meanings. However it is in the nature of mathematics to define whenever possible.

Definition: If there is a number b such that numbers are represented in terms of sums of multiples (up to b - 1) of powers of b, then that number system is said to have base b. If a system seems to have base s up to the number b which is the real base of the system, it is said to have subbase s (see James, and James, **base of a system of numbers**, *tech.;* for subbase, see Closs, p. 3).

I want to return to our own number systems, both verbal and numeral. With the system of number words first, it has a base ten structure up through one thousand. If it is truly a base ten system, we should get another new word for the next power of ten, but instead we get the composite ten thousand. Likewise for the next power of ten we get the composite one hundred thousand. The next new word comes at one million, or one thousand squared. In fact from one thousand on new words are introduced only for new powers of one thousand, and intermediate numbers are composites using multiplicative and additive principles. One is led to conclude that our system of number words is a base one thousand system, with a subbase of ten.

Does the same conclusion apply if we look at our numeral system? Consider the number 4,627,587. One could interpret the commas as serving the same purpose they do in Neugebauer's notation for Babylonian numerals, to separate the positions. In this interpretation the number shows 4 in the millions position, 627 in the thousands position, and 587 in the units position. This interpretation of course matches how we read thenumerals. Thus our numeral system supports the interpretation that we have a base one thousand system with a subbase of ten!

Summary

Numbers are represented in many cultures in two different ways, in words and in numerals. Either way one may talk about the base of the number system, and possibly a subbase as well. Thus the Nengone system of number words has base twenty with subbase five; the Babylonian system of numerals has base sixty with subbase ten. Our systems, both verbal and numeral, which start out as base ten systems, could more accurately be classified as base one thousand with subbase ten.

References

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