From Art and Architecture to Mathematics

From *Intuition* to *Insight.* From *Decoration* to *Demonstration.*

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The difficulty of achieving popularisation without over-simplification or misrepresentation is well known, and will not be discussed here. What follows is an example showing what can be done to provide a non-mathematical audience with some insight into the nature of mathematical discovery in a historical context. This paper is based on a lecture, illustrated by slides, given to a general audience in Melbourne as part of a Council of Adult Education course.

The Origins of Counting

The ancient Egyptians attributed the invention of number, calculation, writing and knowledge generally to *Thoth,* a protector god and patron of order. Hieroglyphic writing, and with it symbols for *one, ten, a hundred,* and so on up to *a million,* was established in Egypt by 3100 B.CE. Even earlier, a quite different form of writing, more abstract, and adapted to pressing marks into moist clay, had arisen in Mesopotamia, around 3400 B.CE. The ability to count, and even to add and to record numbers of livestock with the aid of stones or clay tokens may have long predated writing, going back at least to the beginnings of settled agriculture in the Middle East. Indeed, the *recording* of countings, may well have led to the devising of signs or symbols for the various items tallied, and hence, eventually, to the adoption of a repertoire of conventional forms constituting picture writing.

So counting may have arisen out of the need to establish the size of flocks, or of human populations in order that they might be taxed. Some prehistorians have suggested a possible *ritual origin* in which chanting by participants in processions approaching an altar or other holy place might have produced a *standardized sequence* that could come to be repeated in other situations.

Alternatively, standardized sequences could have arisen with gangs of labourers performing repetitive tasks. This possibility is suggested by surviving work songs and sea-shanties-sung to correlate individual efforts while pulling on ropes, for example or sung just to relieve boredom. Or perhaps names were attached to the succession of days represented by tally marks on scales used as primitive moon calendars. The suggestion is that some specific sequence of words became to be applied more widely, eventually to become the succession of general number words. Compare the recitation by children of "This little pig went to market, this little pig stayed at home..." not only as they touch one toe after another, but occasionally, *and spontaneously,* also in association with other quite different successions of things.

The Role of Art and Architecture

Geometry (literally "earth measurement") may have originated in the need to survey sites as part of any important building project, especially in the age of *megalithic* construction when alignments with sunrise or other astronomical positions were often called for. At the more everyday level, the need to establish boundaries, or-after the Nile floods-to re-survey fields, would have provided the challenge to devise plausible "peg-and-cord" construction recipes for obtaining right angles and so on. Then there is the evidence from decorative art that clearly involved empirical (practical, experimental) explorations of patterns exhibiting various kinds of symmetry. Early artists and artisans responsible for painting pottery, and the walls of temples and palaces, certainly discovered some of the very same patterns that came to be studied by ancient calculators in Babylonia and Egypt, and then, at a more penetrating level, by the great geometers of ancient Greece.

Explanations

The above suggestions are best thought of as complementary-supporting each other rather than as being in competition. It is likely that they all point to activities influential in stimulating the kind of reflection we recognize as *mathematical.* That word comes from the Greek, *mathematikos* = of learning, and the disposition to learn, especially about *determinate* things. *Calculation* and related words come from the Latin, *calculare,* to reckon, originally with pebbles: *calculus,* being the diminutive of *calx,* a stone.

As rituals became more formalised, altars and temples were called for. As rulers became more demanding, palaces and taxation schemes became more elaborate. As the symbolic and aesthetic power of art came to be appreciated, more ambitious projects were undertaken. And more extensive and more intricate constructions necessitated the invention of new surveying methods and calculation procedures.

Setting out a Right Angle

A straight-line crease can be obtained by folding over onto itself a sheet of paper (or papyrus-the writing material in ancient Egypt). Another fold made so as to make the

two parts of the first crease coincide produces a second crease at right angles to the first. Other methods relying on symmetry are indicated in Figures 1 and 2.

The first diagram shows a simple peg-and-cord construction for obtaining *PBP'* at right angles to a given base-line through three equally-spaced pegs placed at *A, B* and C.

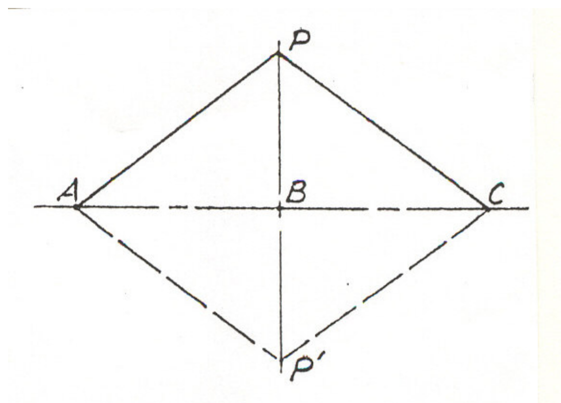
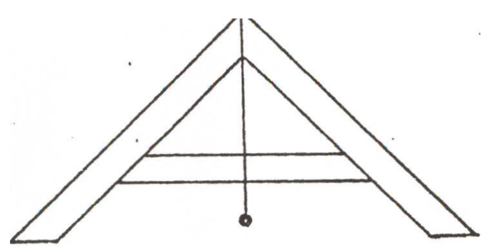


Figure 1

A rope of any convenient length with a mark at its mid-point (found by prior folding over on itself) with its ends fixed at pegs *A* and C is held taut, first to one side of the base-line and then to the other side to locate the positions for pegs P and P' which
define the line perpendicular to *AC* through B.

Figure 2



On their building sites, the ancient Egyptians used the A-frame device shown in Figure 2. This would have given reasonably accurate simultaneous indications of the horizontal and vertical directions. Accurate symmetry of construction, and the alignment of the weighted string over the previously marked mid-point of the cross-piece, would ensure that the two feet were on the same level.

Many centuries later, theoretical versions of these practices (with straight lines in place of cords and frameworks) were examined by the Greek geometers who not merely

described the methods but *proved* their correctness. For example, the key theorem authenticating the construction just described is that which establishes that "the join from the apex of an isosceles triangle to the mid-point of the base is at right-angles to the base." The construction procedures examined by the Greeks were most commonly--but by no means always-those that could be performed using only straight-edge and

compasses.

Art and Perception

The design shown in Figure 3 appears on an Egyptian inlaid wooden box made around 3000 B.CE. It has spread far and wide (and has probably been occasionally reinvented) over the succeeding five millennia.

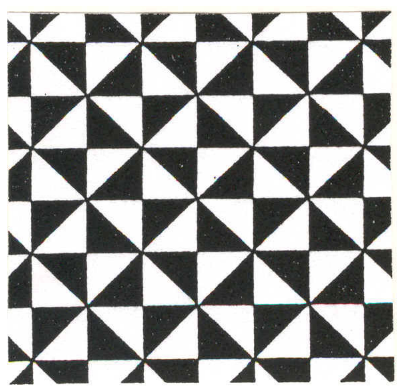


Figure 3

This design deserves more than a casual glance. Possibly it occasioned the first recognition that different observers, or the same observer at different times, may respond differently to a given stimulus. When you look at Figure 3, do you experience a switch from seeing what might be described as clockwise "windmill sails" to anticlockwise ones? An accumulation of experiences such as this would eventually have forced the recognition that meaningful observation is not a passive process. In responding to a present stimulus, we make a remarkably rapid, usually unconscious yet self-convincing contribution-an interpretation. Typically this is in some obscure way dependent on our past experience in relevant situations. And this applies not only in cases of sense perception, but very importantly in the way we "see" a situation involving other people, their motives, the statements they make, and so on.

Different historians presented with the same documentary evidence may understand it very differently because they place it in different contexts, or have different underlying assumptions, interpretations and commitments. Different observers of geometric or
other kinds of art obviously respond in vastly different ways. Different critics make different "readings". What we recognize as fine insight, or intuitive judgment, on the one hand, or prejudice on the other, is dependent on the quality of "the prepared mind". And it is the importance of the prepared mind that is central to appreciating what is often casually regarded as "a chance discovery". This is what education, conditioning and "enculturation" are all about-the preparation of minds!

Eventually, some proto-mathematician might have read from the pattern of Figure 3 the truth that a square having its vertices at the mid-points of the sides of another square has exactly half the area of that other square. And it might also have come to be noticed that there is a simple way of setting out a single square having its area equal to the sum of the areas of two other squares equal to each other-or, conversely, of obtaining two equal squares together equal to a given square (Figure 4).

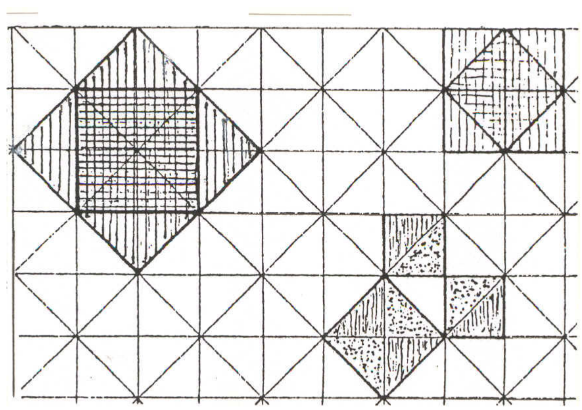


Figure 4

The Art of Mathematical Discovery

The same pattern might also have been seen as providing the implicit challenge, and the opportunity, to find the relation between the lengths of the diagonal and the side of a square. In Figure 5, the two shaded triangles have the same shape. So, if we believed

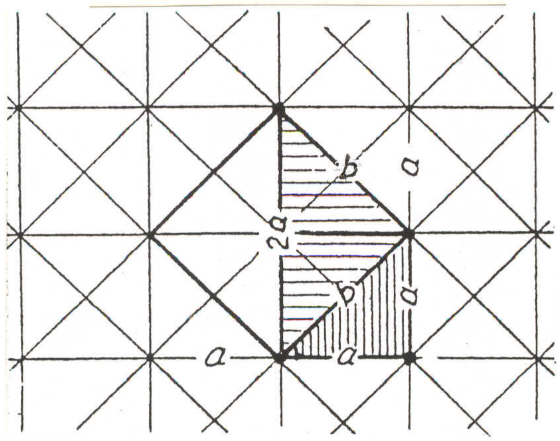


Figure 5

that the diagonal of a square were, say, (1+1/2) times the side length, then both b =(1+1/2)a and $2a$ = (1+1/2) b would have to be true. If we replace b in the second equation with its supposed value of (1+1/2) *a, we* get $2a$ = (1+1/2) times (1+1/2) *a*. But this is false since (1+1/2) times (1+1/2) comes to 2+1/4. Clearly, b is somewhat less than (1+1/2)a. We need a number which when multiplied by itself gives the answer 2. We find that (1+1/3) is too small. It might also be noticed that (1+1/2) times (1+1/3) comes to 2 exactly. The average (arithmetic mean) of (1+1/2) and (1+1/3) is (1+5/12). Is *this* the precise value for the multiplier to be applied to the side length to obtain the diagonal length? No! (Test it by multiplying (1+5/12) by itself.) Can you see how to continue the procedure to get more and more accurate values for the ratio sought?

A clay tablet nearly 4000 years old from Babylonia shows that the extremely accurate value that we would write as 577/408, or 1.41421..., had been obtained for the diagonal-to-side ratio, probably by a continuation of the method just indicated. Extension of the reasoning of the previous paragraph to non-isosceles similar right-angled triangles arranged as in Figure 6 leads to a way of finding the length of the diagonals of any rectangle of given side lengths.

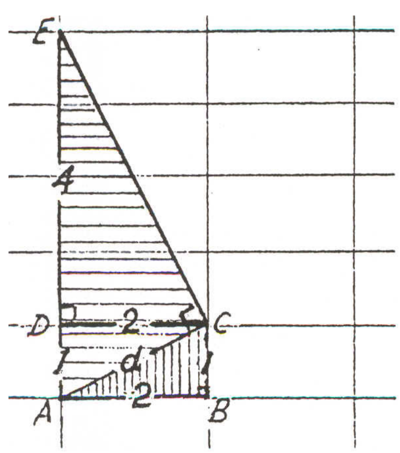


Figure 6

In the particular case shown in Figure 6, notice that right-angled triangle *EDC* has the same shape as triangle *ABC,* the corresponding sides being in the ratio 2: 1. Therefore DE is twice 2 units long. Further, because triangles *ECA* and *ABC* are similar, *AE* (of length 5 units) is the same multiple *(m,* say) of *AC* as *AC* is of *BC*. Hence, *AC* = *m* times 1 unit, where *m* times *m* times 1 = 5. So AC = the square root of 5 units exactly, or a little less than (2+1/4) units- since (2+1/4) times (2+1/4) comes to slightly more than 5. And since 9/4 multiplied by (4/9 of 5) = 5, exactly, the arithmetic mean of the overestimate, 9/4, and the underestimate, 20/9, would yield a good approximation for *sqrt(5),* the true measure of the diagonal of a "2 by 1" rectangle.

From Square Roots to the Discovery of Pythagorean Triangles

Notice that all of the above has been done without supposing that the so-called theorem of Pythagoras was known. However, the systematic discovery of what are known as "Pythagorean" triangles (with sides 3,4,5; 5,12,13; etc.) may be made by just such calculations on variously shaped rectangles as described here in relation to Figure 6. You should satisfy yourself that this is so by starting with a rectangle with sides 3 and 4 units long and show how the diagonal length (exact this time) may be found by the similar-triangles method just illustrated.

A very impressive listing specifying "Pythagorean" triangles has been found on a clay tablet from Old Babylonia, dated to about twelve centuries *before* Pythagoras's lifetime! There is plenty of other evidence showing that there were Babylonian specialists of this period who were highly competent calculators of missing side lengths of adjacent similar right-angled triangles. Other tablets show that the Old Babylonians carried out calculations on diagrams that could have been taken directly from decorative art patterns of Figures 3 and 7.

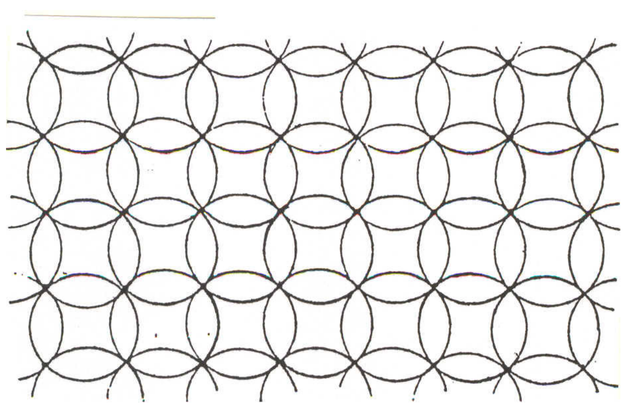


Figure 7

Figure 7 shows a decorative design of remarkable antiquity. It has spread around the world and survives to this day, especially as an all-over pattern on fabrics. The oldest known examples are from the archaeological site of Tell Halaf in Mesopotamia, dating to before 4000 B.C.E. These examples were drawn freehand. But in time they came to be compass drawn, with the centres placed at the intersections in a previously-laid-out grid of squares. Intermediate between the freehand and the compass-drawn examples are those in which the pattern was produced by pressing the end of a hollow cylindrical tool (like a pipe) into moist clay, before drying or baking. Examples of such pressed designs have survived from both Mesopotamia (now Iraq) and the Indus Valley (Pakistan).

The early examples were perhaps taken as representative of a sacred vine, or of the leaves of a tree of life-though the myths associated with these symbols are only known to us from the literate era. Some of the examples on Indus Valley pottery, around 2000 B.C. E., appear to have been painted to look like leaves of the sacred pipal tree *(Ficus religiosa)*-the tree of knowledge or enlightenment. This is the *bodhi-* (= "awakening") or *bo-tree* under which the Buddha was to gain enlightenment. This tree, associated with native earth genii is still revered throughout India. The Indus Valley was the centre of a flourishing *literate* civilization from about 2500 to 1750 B.C.E., but the writing system is now inadequately understood. The Buddha Gautama lived in the sixth century B.C.E.

From Art to Mathematics to Philosophy

The discovery of *derivative* patterns, provide evidence that what might be described as "geometrical readings" of the basic design were made. These are indicated in Figure 8.

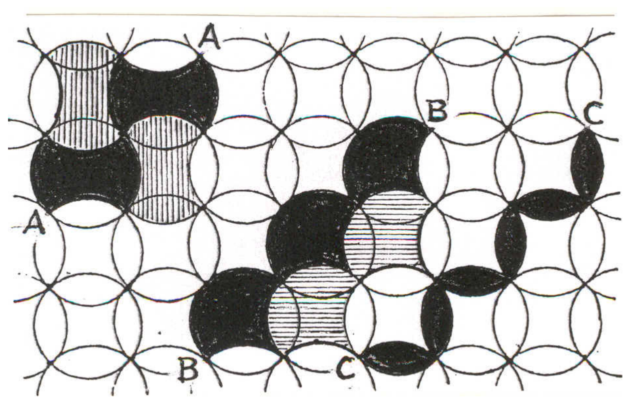


Figure 8

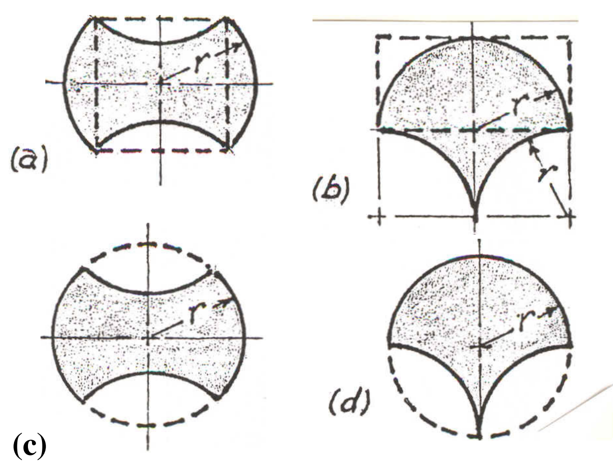
Notice that each of the axe-head or shield-shaped forms, shown at A-A, is equal in area to a square having the same vertices. Perhaps less obvious is the fact that the individual "tiles" which are the elements of the scale pattern, shown at B-B, are equal in area to the shield-shaped forms. Can you explain simply why this must be so? Note that in asking such a question as this, we are not, of course, concerned with inevitable inaccuracies of practical construction. We are dealing with perfect circles imagined to be precisely positioned in relation to perfectly aligned, exactly equal, adjacent co-planar squares! But we never encounter such perfection in this material world of ours. Where then does our determinate knowledge of these things come from? This question became central to the highly influential philosophy of Plato in the 4th century B.C. E., and new myths had to be invented to account for it.

Carefully made baked-brick tiles having the form shown at B-B with coloured glazed faces and set in hard plaster mortar have survived from around late eighth century B.C. E. at Halaf. Freehand painted examples of the same design appeared as early as the third millennium B.C. E. in the Indus Valley. The frieze design indicated at C-C is found on early second millennium pottery fragments from Egypt.

The determination of the area of each of the component leaf- or boat-shaped forms is a much more difficult matter than the determination of the other areas picked out in Figure 8. These other shapes, though bounded entirely by arcs of circles, are readily shown to be equivalent to squares or rectangles of known, or easily determined, area.

Ingenious Simplicity

Figure 9



As indicated in Figures 9 and 10, a simple "give-and-take" procedure-carried out as a "thought experiment"-may suffice to show the equality of the areas. At 9(a), the shield-form is to be seen as obtained from the square by the addition of two equal segments and the subtraction of another two. Hence the equality of the areas of shield and square. At 9(b), the curvilinear form is obtained from the rectangle by the addition of the two equal spandrels below, and this is then exactly compensated for by the subtraction of the two spandrels in the upper part of the diagram. (A *spandrel* is the figure enclosed between a quadrantal arc of a circle and the tangents at the ends of the arc. *Quadrantal* = containing a quarter, from the Latin; cf. *quattuor* = four.)

Further, as illustrated in diagrams 9(c) and (d), the equality of the shield and the fishscale forms is revealed by seeing each as obtained by cutting away two equal boatforms from each of two circles of equal radius. It should be noticed that each of the forms shown in Figure 9 has its perimeter *equal* to the circumference of the associated circle. The area of each is, however, *less than two-thirds* of the area of the circle. Can you work out the fraction more accurately? The branch of mathematics known as *isoperimetry* (from the Greek, *isos,* equal, + *peri,* round, + *metron,* measure) began with just such observations-and responses to the challenge to find the figure having the maximum area for a given perimeter.

The equality of the areas of the curvilinear form and the underlying equilateral triangle, shown in Figure l0(a), *is* immediately obvious. Tiles of this form fit together "to cover the plane", as readily as do congruent equilateral triangles.

Figure 10

(a) *(b)*

The give-and-take principle was applied to several precisely-specified and theoretically-examined crescent-shaped forms called "the lunes of Hippocrates" after the Greek geometer who devised them. This Hippocrates hailed from the Aegean island of Chios. (He is not to be confused with his more famous namesake Hippocrates of Cos, "the Father of Medicine". Both flourished in the second half of the fifth century B.C.E.) One of Hippocrates' lunes is shown in Figure 10(b). This particular lune-form was shown to be equal in area to the indicated quadrilateral having one pair of parallel sides (Such a quadrilateral is called a *trapezium* in British textbooks, and a *trapezoid* in American ones! Both words are from the Greek: *trapezion,* diminutive of *trapeza,* a table.)

A full treatment of this case would involve (i) a justification of the relative side lengths (in this case they are as 1:1:1: - sqrt(3)); (ii) a straightedge-and-compasses construction for this (or any other quadrilateral having one pair of opposite sides parallel) when the side-lengths are specified; (iii) the determination of the positions of the centres, and of the radii, of the two arcs in order that the segments shall have the same shape; and (iv) a further construction to convert the quadrilateral into a rectangle, and thence to a square of equal area. To work through these stages in detail should make a rewarding school-investigation topic.

Proof-the Greek Contribution

What has just been outlined is an example of a classical "quadrature"-namely the finding of a square equal in area to a given figure. Nowadays it is usual to dispense with the construction and to seek a formula (or calculus procedure) enabling the area, etc, to be determined. Either way it is the *proof* establishing that the construction or formula is valid that is insisted on. What distinguishes the pure mathematician from the mere practical calculator is the ability to devise proofs, and the imagination or "intuition" to see what relationships are likely to be provable-and significant in potentially connecting with interesting or important parts of the subject.

Proofs typically depend upon the acceptance of numerous generalizations, which may have been previously proved, or be hypotheses set down for subsequent proof, or they may be axioms-assumptions so basic, that they are at least temporarily assumed, either as being "obvious", *or in order to discover the consequences of supposing them to be true.*

Where, in the outline above, it was stated that the side-lengths were to be as 1:1:1:sqrt(3), and that the arcs be arranged so that the segments shall be similar, this was in order that the largest segment will be equal in area to the sum of the three smaller ones. The chief underlying generalisation is the theorem which may be enunciated as: "The areas of similar figures are to one another as the *squares* of corresponding linear dimensions." A special case, reportedly proved by Hippocrates (who wrote an *Elements of Geometry* over a century before Euclid), is that "the areas of circles are to one another as the squares on their diameters." And *similar* segments are the same fractions of the circles of which they are a part. So similar segments are to one another as the squares on their chords. A segment (from the Latin, *segmentum* = a piece cut off, from *secare,* to cut) denotes, in the present context, the figure bounded by an arc of a circle and the chord that joins the ends of the arc. (Arc is from the Latin, *arcus,* a bow or arch.)

Each leaf- or boat-shaped form picked out along C-C in Figure 8 may be viewed as two equal segments placed with their chords coinciding, one segment being a reflection of the other. As is clear from Figures 8 and 9, the arcs of the segments are quarter circumferences, and the area of each segment is the difference between the area of a quadrant of a circle, of radius *r,* say, and the area of an isosceles right-angled triangle with equal sides of length r units-Figure l1(a).

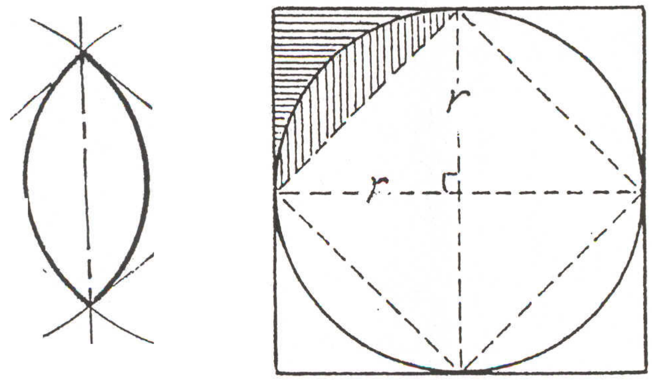


Figure 11

(a) *(b)*

Let us put ourselves in the position of a pioneer investigator who does not yet know how to find the area of a circle (or of its quarter). Figure 11(b) shows the particular segment in a promising context. An unsophisticated viewer might "intuit" that the (vertically-shaded) segment is equal in area to the (horizontally-shaded) spandrel. A consequence of this assumption is that the segment and spandrel each have one quarter the area of the square with sides r units. So the quarter circle would be three-quarters of the square on the radius. Thus the whole circle would have an area of *three* times the radius squared. Apart from the suggestive appearance of the diagram, there are other considerations which might have been felt to support this result, and the assumption that led to it.

Extra-mathematical Influences

In ancient times, the square was widely regarded as a symbol for the Earth, and the mundane, in contrast to the circle symbolizing the heavens and the divine. The influence of this belief is clear in Indian mathematics where constructional recipes were worked out for "the circulature of the square." Apparently, these originally played an important part in purifying rituals, which required the construction of altars of circular and other precisely prescribed geometrical forms. In China, a similar belief system inspired the construction of the great, *triple-terraced, circular* Altar of Heaven and associated buildings on the south side of Beijing. And on the northern side of the city stands the great *square* Altar of Earth, with accompanying temples and treasuries. The time of the great sacrifice at the southern altar was the winter solstice (corresponding to the shortest day). The great sacrifice at the northern altar took place at the summer solstice.

There is no doubt that such beliefs-central to the whole world outlook in earlier times- influenced the kinds of explanation that "naturally" occurred to the early natural philosophers in Greece and elsewhere. The Pythagoreans are especially noted for having combined number mysticism with genuine mathematical achievement. The most primitive form of explanation consists in the assumption of *correspondences* between different things and events- for instance between the "indications" or "influences" of the planetary motions, or the risings or settings of the stellar constellations, and the fortunes of individuals or clans or nations.

Even the great natural philosophers, until around 1600 C.E., assumed that the planetary movements must be reducible to combinations of uniform *circular* motions, since the circle was the "perfect" geometrical figure. It was still thought that numerical and geometrical *harmonies* must prevail in the heavens, if not here on earth. What then more appropriate than that the circle, as the symbol of eternity, perfection and harmony, should have exactly *three* times the area of the square on the radius, and that the circumference is three times the diameter? A discussion of the conjectural prehistory of the falsification of this pair of assumptions, and the history of the determination of more and more accurate approximations for the ratio which came to be written as “pi”, must be left for another occasion.

John Pottage