What "Is" Mathematics?

In Memoriam of
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It is an open secret among scientists that accurate predictions made on the basis of scientific laws are rare. Only yesterday, in the dark age of Carnap and Reichenbach, prediction was believed to be the fundamental feature of science. This unrealistic belief went hand in hand with a preposterous confusion, namely, the identification of all of science with physics. In that age of savage simplification, shining examples of prediction in science were the confirmations of the special theory of relativity, the explanation of the spectral lines of the hydrogen atom by quantum mechanics, and sundry other pillars of progress gleaned from the science of mechanics.

It took philosophers well over fifty years to carry out a reality check on their philosophy of science and to diagnose the normative disease that has plagued philosophy in this century. A casual inspection of any science other than physics or chemistry proves beyond reasonable doubt that scientific prediction in the sense of the positivists is at best a cruel joke. Zoology and cosmology, economics and evolutionary biology are only tangentially concerned with accurate prediction.

A description of the scientific enterprise pruned of normative presuppositions lies still in the future. Meanwhile, we may begin to chip away at the barriers that stand in the way. One such barrier is the systematic misuse of language by philosophers and logicians. Common words are rudely deprived of the multiple and contradictory meanings that they enjoy in ordinary language; after the straitjacket of a fixed meaning for every word is imposed, the door is shut to realistic description.

We have chosen the word "is" as paradigmatic of the constipation of meaning from which contemporary philosophy is suffering. We describe some of the multiple senses of the question: "What is mathematics?" when the question is asked in various circumstances.

The reigning orthodoxy of philosophy identifies the uses of "is" with the restricted uses of the word "is" in Fregean logic. Logic has achieved in this century a state of perfection that few mathematical theories have matched. However (or perhaps for this very reason), logical reasoning has become totally divorced from actual reasoning, the kind that is found in real life. In most worldly circumstances, logic shines by its absence. A compelling logical argument is the last weapon of the rhetorician, a recourse to be appealed to in desperate situations, when all else has failed.

It is no accident that substantial applications of Fregean logic are found daily in computer science. The crazies of the eighties, who pretended to simulate the mind with primitive computers, have succeeded, by a display of illiterate reductionism, in clearing up the abyss that separates human discourse from logical deduction.

It is thus no surprise to realize that the meanings of the word "is" prescribed by logicians are a lot closer to the ranting of Ayerian philosophy-fiction than to the richness of senses of the word "is" in everyday writing and conversation.

The accusation of being "illogical" may be leveled at us. Our retort will be a call to duty: realistic description is a paramount task of the philosopher. The first step in a philosophical description consists in admitting that the real is seldom rational and the rational is seldom real. An "Abgrund" separates "Verstand" from "Vernunft." Philosophical description must grapple with open-ended varieties of irreducible cases, with contradictory and ambiguous conclusions which Enlightenment Reason has ignored.

It is our contention that the word "is" in the question "What is mathematics?" does not have a classifiable set of meanings. This contention in no way implies that the word "is" is devoid of meaning. Quite the contrary: we are confronted with an "embarras de choix" among the meanings of "is."

What follows is a partial list of contexts in which the question "What is mathematics?" is found. The list is deliberately biased; it is meant to lead up to a conclusion decided upon in advance.

"IS" AS DICTIONARY DEFINITION

Literally, the question "What is mathematics?" calls for a "definition" of mathematics.

We have been trained to restrict the meaning of the word "definition" to the role of definition in axiomatic mathematical systems. This mathematical sense of the word "definition" will be henceforth disregarded. The senses of the word "definition" in ordinary discourse bear little relation to mathematical definition.

> *The skeptical argument mistakenly concludes that no definition can be "given" in the dictionary, except in the eyes of the beholder.*

The word "definition" occurs in a great many vague and unclear senses, which it would be presumptuous to list. The most common, as well as the most ambiguous is the "definition" that we expect to find when we look up a word in a dictionary.

What is a dictionary definition? In what sense do dictionaries "define" words?

An old skeptical argument purports to prove that dictionary definition is impossible. It runs roughly as follows. Suppose you look up the meaning of word A. The dictionary explains the meaning of A in terms of words B, C and D, say. It may happen that B, C and D categorically specify A as the sole word satisfying certain conditions. But this happens very seldom. More frequently, the dictionary explanation of A in terms of B, C and D is likely to be a vague approximation to the meaning of A. The reader is asked to "get a feeling" for A by various tricks: the explanation of A in terms of B, C and D may be the description of a general class of which A is a member, or it may be a list of likenesses, of comparisons with other objects that are meant to be "like" A; one reads various indirect hints to the meaning to A. What cannot be given in a dictionary is "the meaning" of A.

This frustrating remark by no means implies that the reader will miss the meaning of A when looking up A. The reader is expected to grasp the meaning of A by letting his imagination roam "beyond" the various statements in the dictionary that are meant to "lead up to" the meaning of A. The meaning of A can be grasped only when one looks "away" from the dictionary explanations "towards" some other sense that is not given there, but which the dictionary explanations "point to." No amount of explanation can make sure that the reader will take the leap that will disclose the meaning of A.

The skeptical argument mistakenly concludes that no definition can be "given" in the dictionary, except in the eyes of the beholder.

The preceding argument stands in contrast with what actually happens when someone looks up a word in the dictionary. In point of fact, people do find the meaning of words in the dictionary. If we look up the word "jaguar," we will get an adequate idea of a jaguar, even if we have never seen a jaguar or know nothing about jaguars. When I look up the word "chair," I get a pretty good idea of what a chair is, even if I do not quite grasp the full meaning of what I have "read" until I become familiar with actual chairs.

Dictionaries of synonyms are a further confirmation of the same phenomenon. One learns the meaning of a new word from approximate explanations, by a process that cannot be rationally accounted for. When we look up the synonyms of a known word, we are searching for some word that we may never have seen. We approximately guess the meaning of the synonyms, even though these meanings are nowhere given, and we select an appropriate synonym which we may never have previously seen or used.

The non-rational grasping of the meaning of a word from approximate explanations is an instance of the phenomenon of the "copula," of the function of "is" in "A is B." What matters here is that the "is" acts as a copula only for certain A's and B's, like "jaguar" and "chair." Certain other words pose a different problem of "is" that is not subsumed in the "copula" sense of "is." One such word is "mathematics." Any "mathematics is..." sentence given in response to the question "What is mathematics?" will be evasive. No sensible dictionary definition of "mathematics" can be given.

"IS" AS INVITATION

The situation is different when we look up the word "mathematics" in an encyclopedia rather than in a dictionary. In an encyclopedia we find summaries of entire mathematical fields, as well as a bird's eye view of various branches of mathematics and an ample bibliography that will guide us to learning mathematics.

Is the description of "mathematics" in an encyclopedia an adequate answer to the question "What is mathematics?" unlike the dictionary? Clearly not. The explanation of "mathematics" that we find in an encyclopedia skirts the question by referring us, after an enticing preamble, to technical expositions and classical treatises.

Both the dictionary "is" and the encyclopedia "is" are motivated by the widely felt need of explaining esoteric words in exoteric language. This need roughly dates back to the Renaissance, when the first dictionaries (in the contemporary sense of the term) were compiled. Throughout history, notably in the Middle Ages, no need for exoteric expositions was felt. Explanations (often labeled "definitions") were an internal affair for specialists, from which the public was excluded. The scholastic definition of "Deus" as "eng perfect-issimum" was meant to be shared by philosophers and theologians only. Uttering such a statement in the course a Sunday sermon at Mass might have led to an accusation of heresy.

The Renaissance-Enlightenment notion of definition as exoteric explanation is motivated by the democratic ideal of a universal culture. Such a laudable objective does not make exoteric explanation any easier. Fortunately, an exoteric explanation of mathematics is seldom what the questioner expects when posing the question "What is mathematics?". Let us see.

"IS" AS COPOUT

The question "What is mathematics?" is often asked when the questioner has little or no acquaintance with mathematics, and wants to discharge his or her duty to learn something about mathematics, hoping for a short answer.

The question "What is mathematics?", asked to a mathematician by a person ignorant of mathematics, makes mathematicians uneasy. The mathematician senses dishonesty in the abruptness of the question. The questioner believes that an answer can be given, similar to the answers one can give to questions like "What is boeuf bourguignon?", "What is yellow fever?" or "What are Magli shoes?".

The questioner does not want to learn any mathematics when he asks the question "What is mathematics?". The opposite is true: the questioner wants to rid himself of the need of learning any mathematics whatsoever. He wants to add to his conversational repertoire some brilliant answer that will permanently excuse him from any further dealings with the subject.

One cannot escape the duty of giving a nutshell answer to the question "What is mathematics?", despite the dishonesty of all short answers. Non-mathematicians need to have some idea of what mathematics "is" without having to study mathematics. They are dealing with mathematics as outsiders, but their dealings will affect the future of mathematics: mathematics requirements for schools must be determined by professional educators; mathematical proficiency among employees in a firm has to be gauged. Worst of all, the allocation of research funds for mathematics is made by individuals who have at best a fleeting acquaintance with the subject. Mathematics, like all intellectual disciplines, is not economically self-sustaining , and since the beginnings of civilization mathematicians have depended for their survival on the largesse of society or of a few wealthy individuals. Mathematicians, like philosophers and artists, are "kept" persons. In return, the public expects mathematicians to make the results of their work accessible to cultivated persons who may have a passing interest in mathematics, or who deal with the political and economic problems of mathematicians.

We will leave to another occasion the tragedy that has resulted from the mathematicians' failure, going all the way back to Pythagoras, of giving exoteric accounts of their field that the public could appreciate.

An accessible and short answer to the question “What is mathematics?” may be difficult to give, it may turn out to be dishonest and inadequate, but the mathematicians’ failure to provide such an answer has been a costly mistake.

“IS” AS ESCAPE

Students confronted with the task of learning a mathematical theory rarely feel the need to ask the preliminary question “What is mathematics?”. They are more likely to ask specific questions, such as “What is topology?”, “What is the Riemann hypothesis?”, “What is a random variable?”.

Are we to infer that no answer to the question “What is mathematics?” can ever be given?

Suppose nevertheless, by way of thought experiment, that a student of mathematics were to ask such a question, on the basis of his claim that an answer to the question is a condition to be met preliminary to his getting down to serious study.

It is likely that a teacher hearing such a question from a student would give the student a strange look. The teacher would be put on guard: is the student unfamiliar with grade-school mathematics? is the student afraid of learning mathematics? does the student believe that an authorization is to be granted before undertaking the study of mathematics? is the student afraid of mathematics? does this student require medical attention?

In each of these instances, the teacher will not hazard an answer to the question. Most likely, the teacher may whisper to the student a few soothing words, not in the least meant to provide any explanation of what mathematics is, but rather meant to allay the anxieties that the student’s question betrays.

“IS” AS SUMMING UP

Some mathematicians who are reaching the end of their careers (Poincaré, Hadamard, Weyl), feel the need to answer the question “What is mathematics?” as a prop to their fading hold on the subject, much as they might feel the need to write an autobiography. In these circumstances, the question “What is mathematics?” is an excuse for excursions into the history and philosophy of mathematics. The essays written in answer to this rhetorically posed question will deal with the “nature,” the “structure,” the “standing” of mathematics. The “is” is once more skirted by being turned into an “about,” into discussions about the mathematics of the time, about future directions of mathematics, about relationships among various fields of mathematics.

“IS” AS IMPOSSIBLITY

We have argued that no answer to the question “What is mathematics?” can be given in the form “mathematics is...” by examining some contexts in which the question is asked. In none of the instances considered can the question be given an answer of the form “mathematics is...” In the first instance an answer of the form “mathematics is...” may be read in a dictionary, but such an answer is not taken seriously.

Are we to infer that no answer to the question “What is mathematics?” can ever be given?

Let us call a word X a “pre-ontological term” whenever no adequate answer to the question “What is X?” can be given in the form “X is...“ The preceding examples suggest that “mathematics” is a pre-ontological term.

Most words of common usage are not pre-ontological terms. For instance the word “chair” is not a pre-ontological term, since we can answer the question “What is a chair?” by sentences of the form “a chair is...” An adequate such set of sentences will provide a description of chairs that is good enough for most purposes, even though no set of sentences may succeed in “defining” the word “chair” in the logical sense. We use the word “item” to denote any word X for which an adequate (though not necessarily logical) answer can be given to the question “What is X?” in the form “X is...” “Chair,” “triangle,” “jaguar” are items. Our claim is that there are pre-ontological terms, and pre-ontological terms are not items.

The philosophical literature is rich in pre-ontological terms: “time,” “world” and “nothing” are three pre-ontological terms that have been studied in the phenomenological literature. The question “What is time?” has been deemed unanswerable by philosophers since St. Augustine. The question "What is 'nothing'?" is obviously intractable. In Chapter 3 of *Sein und Zeit*, Heidegger argues that "world" is a pre-ontological term.

The limitations of the language by which we describe and define items, a language made up of "A is B"-type sentences, stand in the way of describing (let alone defining) pre-ontological terms. One is forced to choose between two alternatives: either to decide that no sense can be made of sentences of the form "X is..." whenever X is a pre-ontological term, or else to find a sense of the word "is" that is distinct from the "is" as copula in the language of items.

The first alternative was followed in phenomenology, by an argument—the joint work of several authors—that we will try to sketch.

What "common" features of the words "mathematics," "time," "nothing" and "world" lead us to classify these words under the heading of pre-ontological terms?

The word "is" used as copula in "A is B" presupposes a context of sense-making. A can only be B within a background of unthematized features that are ordinarily passed over in silence. More formally: the "is" of "A is B" presupposes a context within which the "is" can "be." For example, "chair" presupposes a context of everydayness in which chairs are useful. No item can "be" without some such background context. "To be" is "to be-in-a-context." We read, pronounce, deal with the sentence "A is B" while pretending that the meaning of the sentence is to be found "in" the sentence itself, independently of any contextual background. The canons of logic foster the pretense of a decontextualized meaning of "A is B"; the cliché sentences given as examples in logic textbooks are carefully cleansed of contextual references. One can hardly imagine any such sentences ("the snow is white") ever used in daily conversation. But whenever "A is B" is used meaningfully, i.e., contextually, an unthematized background context can always be brought to the fore.

The "is" in "A is B" purports to explain A in terms of B. Such an explanation is made possible by a multilayered twine of contextual and intercontextual senses that link A to B. Without such an underlying contextual/intercontextual twine, no sense can be made of "A is B."

The "is" of "A is B" is meaningful if both A and B are items, i.e., whenever both A and B are ensconced in a common context. However, the statement "A is B" becomes problematic when either A or B are pre-ontological terms. Pre-ontological terms are not items, but conditions of possibility of the contextuality that allows items to "be." In plain words: no sentence of the form "time is..." can make sense, because "time" is not an entity of any kind, but a condition of possibility of all entities.

However, the impossibility of making sense of any "time is..." sentence does not deliver us from the problem of understanding the pre-ontological phenomenon of time. Rather, it points to the need for a language other than the language of items that will be suitable for the inquiry into the sense of time.

No "definition" of the term "mathematics" can describe that particular context that we call mathematics. Mathematics is not an item that certain contexts share. Mathematics is the condition of possibility of mathematical contexts. We cannot explain what mathematics "is" by sentences of the form "A is B," where A and B are items, because mathematics "is" no-thing.

The word "is" is misused when we try to explain what mathematics "is" in the language of contextual items. Questions like "What is mathematics?", "What is time?", "What is the world?" are misleading. Mathematics, time and world are not items, and hence it makes no sense to ask what they "are."

"IS" AS A WONDER

Are we to conclude that the question "What is mathematics?" should be dismissed as meaningless? Such a conclusion would be strikingly similar to the anathemas of the positivists, always ready to liquidate as "meaningless" any question beyond the reach of their narrow vocabularies. Besides, such a conclusion would bring back the specter of normative philosophy from which we have proudly distanced ourselves.

The question "What is mathematics?" is not always asked by way of a copout, as in the examples above. The question "What is mathematics?" is sometimes posed, both by the student and by the mature mathematician, to express a feeling of wonder, to signify the estrangement that possesses us at times, the same estrangement that is felt in the contemplation of the starry sky and the moral law, described by Kant at the beginning of his “Critique of practical reason.” This feeling of estranged wonder is the opening to philosophical inquiry, as Aristotle was first to note. The question “What is mathematics?” may express the feeling of the wonder at the contemplation of the awesome edifice of mathematics.

The feeling of wonder that is sometimes expressed by the question “What is mathematics?” is not likely to be an “answer” to the question “What is mathematics?” It will be the start of a philosophical journey that will eventually disclose of the “conditions of possibility” of mathematics. The disclosure of such conditions of possibility is the “answer” to the question.

Sadly, philosophers have neglected the task of giving a rigorous formulation of the method of reasoning that leads to the disclosure of conditions of possibility. If the day ever comes when the “logic” of conditions of possibility, i.e., philosophy, is developed with the standards of rigor that have been set by Fregean logic, then an “answer” to the question “What is mathematics?” will be possible in the form “mathematics is ...”

Hotel Infinity

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May be sung to the tune of the Eagles’ “Hotel California” by Don Felder, Don Henley and Glen Frey.

On a dark desert highway—not much scenery
Except this long hotel stretchin’ far as I can see
Neon sign in front read “No Vacancy,”
But it was late and I was tired, So I went inside to plea.

The clerk said, “No problem. Here’s what can be done—
We’ll move guest in room N to the next higher one.
That will free up the first room and that’s where you can stay.”
I tried understanding this as I heard him say:

Chorus:
“Welcome to the Hotel In...finity—
where every room is full (every room is full)
Yet there’s room for more.
Yeah, plenty of room at the Hotel In...finity—
Move ‘em down the floor (move em’ down the floor)
To make room for more.”

I’d just gotten settled, I’d finally unpacked
When I saw 8 more cars pull into the back.
I had to more to room 9; others moved up 8 rooms as well.
Never more will I confuse a Hilton with a Hilbert Hotel!

My mind got more twisted when I saw a bus without end
With an infinite number of riders coming up to check in.
“Relax,” said the nightman. “Here’s what we’ll do:
Move to the double of your room number:
that frees the odd-numbered rooms.”

Repeat Chorus

Last thing I remember at the end of my stay—
It was time to pay the bill but I had no means to pay.
The man in 19 smiled, “Your bill is on me.
20 pays mine, and so on, so you get yours for free!.”

A Geometry Course for Prospective Secondary School Teachers

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High school graduates of the near future could be more sophisticated geometers than their professors. If the National Council of Teachers of Mathematics Standards [7] is adhered to, college-intending high-school students will learn more than basic Euclidean geometry. They will have worked with technological aides to make discoveries and then made deductive arguments to verify their conjectures ([7], 159) and spherical geometry ([7], 160). They will have developed both deductive and inductive reasoning in part by exploring geometry through short sequences of axioms. Students entering college will have an "... appreciation of Euclidean geometry as one of many axiomatic systems" ([7], 160).

That is, of course, if their high school teachers know the material and can teach it effectively using the pedagogical techniques suggested in the NCTM guidelines ([7], [8]). The NCTM expects high school teachers to introduce their students to spherical geometry, software programs, axioms, and deductive reasoning and proofs. In the classroom they are expected to incorporate tasks that require students to make and test conjectures and use manipulatives as well as to have students work in cooperative groups. Moreover, they will be expected to use a variety of methods to assess the student's progress in the course.

The geometry course described in this note is an attempt to help prepare prospective teachers to meet the goals described in the Standards. Some recent literature confirms that teachers teach material in the way they were taught and that, to make effective use of a pedagogical technique, it helps to have learned the material by the same technique ([1], [2], [6]). With the above in mind, we replaced the traditional "Foundations of Geometry" course with one that incorporated technology, discovery learning in a cooperative-group setting, and introduced students to non-Euclidean geometry early in the course.

The "Foundations of Geometry" course we replaced exhibited many features common to the majority of the geometry courses offered to prospective high school teachers in the United States [5]. It was taken primarily by prospective teachers, lecture-based with some group work, and made little use of manipulatives and software tools. In our course, like in about half such courses, the material was developed following Hilbert's or Birkhoff's axioms. First, absolute geometry was developed and then, towards the middle of the course, the parallel postulate was introduced. At that time the students were introduced to some spherical geometry and hyperbolic geometry. The course then returned to Euclidean geometry to develop similarity, area and properties of the circle.

For instance, that the angle sum of a triangle in Euclidean geometry is 180° appeared as follows. The first half of the course developed the incidence, betweenness, and congruence theorems and geometric inequalities. The elliptic, hyperbolic and parallel postulates were then introduced along with models of the different types of geometries. Following this, using Saccheri quadrilaterals, it was shown (as a theorem of absolute geometry) that the angle sum of a triangle is less than or equal to 180°. It was then shown (without recourse to the result from absolute geometry) that in Euclidean geometry the angle sum of a triangle is 180°. This struck the students as quite a bit of work to get to something they "already knew." A more serious difficulty with this approach was that there was little to challenge the students' high-school-based knowledge of Euclidean geometry before hyperbolic geometry was introduced. Hyperbolic geometry was usually introduced several weeks into the course and then (nearly) abandoned while similarity and other Euclidean topics were studied. In addition, as the proofs appeared to be merely confirming what the students felt they already knew, the proofs did little to promote a deeper understanding of the material.

We replaced our traditional course with one that forces students to confront what they 'already know' early in the course. With the help of tools that secondary teachers will eventually use in their own teaching, the course creates a need for axioms and proofs to describe and work with different geometries. The new course currently develops the topics using cooperative group projects in the following order: area, the angle sum of a triangle and the parallel postulates, congruence of triangles, similarity of triangles, properties of circles, and transformational geometry. During these projects the students use tools such as the Geometers' Sketchpad, Non-Euclid (a software program that models the Poincaré disc), the Lenart sphere (a clear plastic sphere with a spherical protractor and compass), a MIRA (a plastic device that acts as a mirror to do reflections), Geoboards and, of course, a compass and straight edge. Material that cannot be easily introduced in the projects is introduced in lectures and used for individual homework assignments. For instance, the betweenness axioms are introduced along with the exterior angle theorem in a lecture. About 70% of class time is spent with the students working in cooperative groups with the remainder of the time being used for lectures and exams.

Each project consists of two or three subprojects that each require a written 'progress report,' and conclude with the preparation of a final report that has the students synthesize the findings of the subprojects and correct any errors that appeared in the progress reports. The progress reports and final reports form the basis of a written dialog between the students and the instructor.

A short discussion of the first two projects shows how the assorted elements of the course fit together. The first project has the students develop a theory of area. The first subproject asks them to develop a procedure for finding the area of a polygonal region assuming they know how to a) find the area of a square and b) find the area of a triangle. Students then use these procedures to justify the standard formulas for the area of a rectangle, parallelogram and trapezoid. This work is followed by a lecture on axiom systems and models. The second subproject has each group develop a set of area axioms and then use the axioms to prove their formulas from the first subproject. These axioms are also discussed as a class. The third subproject has them use the Lenart sphere to test the validity of their axioms on the sphere and derive a formula for the area of a spherical triangle. The final report has them integrate the (corrected) results of the subprojects into a single document. As part of their work, the students are asked to identify any apparent gaps or holes in their arguments, for instance, any assumptions that they are making about length, the area of a boundary, et cetera.

The second project addresses the angle sum of a triangle. The students are first asked to develop a system of axioms that allow them to prove that the sum of the measures of the angles of a triangle is 180°; this usually requires an axiom stating that alternate interior angles are congruent. In the second subproject they explore the validity of their axioms and the exterior angle theorem on the sphere and in the Poincaré disc using the software program non-Euclid. This progress report also requires each group to make a conjecture about the area of a triangle in the Poincaré disc. In the final report the students show that the Euclidean parallel postulate in conjunction with the exterior angle theorem yields that the angle sum of a triangle is 180°. During this project there is a lecture on the history of the parallel postulate and the development of non-Euclidean geometry. (The angle sum result for absolute geometry is proven later in the course.)

At this stage the students are 6 to 7 weeks into the course. The students have had significant exposure to spherical geometry and the Poincaré disc. They have been surprised to discover that the area of a triangle is not always 'half the base times the height' and that the angle sum of a triangle is not always 180°. The students go on to explore the congruence of triangles, similarity and transformational geometry in Euclidean and non-Euclidean geometry. In the last two projects the students investigate geometries through the fixed points and lines of reflections and classify motions in the plane.

In the early part of the course the students develop their own axioms and lemmas for each project. As the course progresses, to maintain some uniformity in the axiomatic development, assorted key 'axioms' are suggested to them; for instance, in the angle sum project they are given the exterior angle theorem as an axiom, and later in the course they establish it as a theorem. Eventually, during the last two projects on transformational geometry they are given the definitions and axioms they need for each subproject and are asked to prove a variety of theorems. These projects have them work extensively with the assorted software tools and manipulatives that were introduced earlier in the course. The definitions and axioms are given without intuitive motivation or explanation; it is up to the students to 'discover' the intuitive content of the definition through the models developed during the course. For instance, the students are given the definition of fixed points and fixed lines of a motion and then, to help develop their intuition, are asked to find the fixed points and lines of reflections on a sphere and in a model of Euclidean geometry.

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Most students benefited from this new course structure. From our observations, we concluded that students improved in their ability to discuss mathematics, explain their mathematical thinking, and work with others toward a common goal. On course evaluations students reported that they deepened their understanding of geometry, that the group work and computer software facilitated their understanding, and that they increased their self-confidence to do geometry.

There are some drawbacks to the course. One is that, as we implemented it, it requires quite a bit of classroom time; to accommodate this we added a weekly two hour lab to the course. As one would hope, we were able to investigate the standard topics and address additional concepts in the replacement course; in particular, transformational geometry was explored in much greater depth than in the traditional course. It is possible that the same amount of material could be investigated in a course with fewer contact hours by having the groups do some work outside of class. In our course most groups were able to do most of their group work during class.

Another drawback is that since the students are developing the axioms in a nonstandard order, at least two different axiom systems are introduced, and what is an axiom one week may become a theorem the next week, the students have some difficulty in seeing different sets of geometric axioms as coherent systems. The instructor needs to monitor the groups closely to prevent errors due to improperly blending axiom systems. For instance, the first two times the course was taught transformations were introduced through MIRAs and, as a result, the students tended to assume that reflections exhibited all the properties of reflections in a Euclidean plane. Even after working with reflections on a sphere, students still slipped into making assumptions based on their initial work in the Euclidean plane. Some of these problems can be avoided by making the assumptions in the projects very explicit. Even though there is occasionally some confusion during the course, in the end it is worth the extra vigilance to help the students develop the perspective needed to appreciate the role of axioms in mathematics.

A drawback of a more mundane nature is that the course can be very time consuming for the instructor. Each group submits ten to twelve written reports during a quarter. As the subprojects build on one another and are used to prepare the final report, they need to be graded promptly and carefully. Homework and exams also need to be graded. In addition, organizing the class into cooperative groups requires the instructor to do more administrative work than a lecture-based course.

A possible philosophical objection is that much of the grade is based on cooperative work. Since cooperative group work plays a large role in the course, group grades constitute a significant portion of the individual student's final grade. In our courses thirty-five to forty percent of the final grade was based on the cooperative projects. Consequently, it is important to be sure that the group grade reflects the sum of each individual's understanding of the material. One way to help accomplish this is to give individual quizzes at the conclusion of a project and make the combined group score on the quizzes part of the group grade. According to journal entries and (anonymous) student evaluations, during our courses the students' attitudes regarding group grades changed from some concern that some students would get undeserved credit to a general belief that they were fair.

Even though the course was designed with the needs of prospective secondary teachers in mind, the course is also appropriate for a mathematics student. It is, at its core, a mathematics course. Except for a brief discussion of cooperative learning and group work (20 minutes), no class time is spent discussing pedagogy. Since the course gives the student the experience of mathematical discovery and actively learning mathematics, we believe it benefits the typical junior level mathematics major and is a viable replacement for a traditional "Foundations of Geometry" course.

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Tryst of Twins: Antarctica, Amazon

Arnold L. Trindade
Glen Cove, N.Y.

How crystal white the ice cap Neptune head
Views the ocean; streaming ice waters beneath
Are lubricant carrying his body
Gliding steadily to the sea.

His equatorial giant twin, the Amazon,
Suckling the breast of dark rain clouds
Transfuses oxygen, a bloody, muddy flow,
The umbilical for starving embryos, millions.

A biopsy of the ice cap reveals
Microbes fungal, bacterial species,
As do probes in Amazon's forest hair:
Nesting plant, bird, lichen fair.
Who might guess 'neath the Atlantic deep
Antarctic waters meet unseen in tryst?

In kisses hugging, bedside currents, embraces
The Amazon body in earth's one living womb?
While the surface conflicts, retards proliferation
Of stagnating antigen-antibiotic abcesses
Deep under spherical transfusing blood says
Planet love, is, such flowing expecting no return.

Haiku: The Heart

Arnold L. Trindade
Glen Cove, N.Y.

Big Bang disperses
Heart rub-a-dub calibrates
How in tune palpitates

Naïve Thoughts on the Paradox of Gödel

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EPIGRAPH

You can't get there from here.

The classic Sam Loyd "Fifteen Puzzle" consists of fifteen movable and numbered square counters placed in a random order in a four by four square frame. One is allowed to slide a counter into an empty space, and the goal is to arrive at the natural ordering of the counters from a given initial arrangement by a sequence of such slides.

Theory shows that starting from half of the possible original positions of the counters, the puzzle is solvable while from the other half it is insolvable. But a simple interchange of any two counters will alter the puzzle from solvable to insolvable or vice versa. Embedding the board in three dimensions makes the puzzle always solvable.

***

THE IMPETUS

The impetus for this paper came from a novel by Apostolos Doxiadis, *Uncle Petros and Goldbach's Conjecture*. In the novel, Petros, a mathematician, has set his heart on solving the Goldbach Conjecture. Failing and deeply disappointed, Petros takes psychological refuge in the possibility offered by Gödel's Theorem, that the problem be insoluble. After reviewing this novel for the *SIAM NEWS*, I began thinking about my own reactions over the years to the Gödel Theorem and to one particular aspect of it.

ON THE WORD "NAÏVE" IN THE TITLE

The amount of material related to Gödel's Theorem is enormous: it is far beyond anyone's ability to know it all or understand it. And the Web has multiplied the chatter and blurred it into an incoherent mass of thoughts.

I will approach my topic as I have experienced it in my professional career as an applied mathematician and as a writer. This is the meaning of the word "naïve" in my title.

WHAT IS THE PARADOX?

Simply stated, the Paradox of Gödel is this: although Gödel's Incompleteness Theorem has been touted in some quarters as the most significant mathematical achievement of the 20th century, it seems to be of little significance to the bulk of research mathematicians. Why is this the case?

What is Gödel's Incompleteness Theorem? In nontechnical language one might say: if a mathematical statement has been asserted that seems to make sense, it may not be possible to prove whether the statement is true or false.

Example: A claim has been made that there are an infinite number of 0's in the decimal expansion of π. At the moment, it is not known if this is true or false or if it is undecidable.

I like to put the incompleteness theorem this way: given a mathematical "there" and a "here," you may not be able to get there from here.

Slightly more technically: arithmetic is not completely formalizable. For every consistent formalization of arithmetic, there exist arithmetic truths that are not provable in that system.

"There will always be arithmetic truths that escape our ability to fence them in use the tools of rational analysis" (John Casti).

In what follows, I will use the abbreviation GIT to designate the Gödel Incompleteness Theorem, or more generally, any of its closely related theorems or equivalent formulations.

THE ROMANCE, THE HYPE, AND THE ICONOGRAPHY SURROUNDING GÖDEL'S THEOREM

"An astounding and melancholy revelation" (Ernest Nagel and James R. Newman, *Gödel's Proof*, 1958).

"The most decisive result in mathematical logic" (Boyer, *A History of Mathematics,* 2nd ed.).

"Amazing, shattering" (Morris Kline, *Mathematics: The Loss of Certainty*).

"Mind boggling. One of the pinnacles of human intellectual achievement. Basis for a whole host of related developments in philosophy, computer science, linguistics, psychology. Mankind will never know the final secret of the universe by rational thought (Casti, *Reality Rules*, *II*).

"Only Einstein's theory of General Relativity represents an accomlishment of comparable intellectual grandeur" (Berlinsky, *Black Mischief*).

"GIT a part of a 'golden braid' of math , art and music that penetrates the very nature of human consciousness. Gödel-numbering has opened up vast new worlds" (Douglas Hofstadter, *Gödel, Escher and Bach).*

One can find statements in semiotics, theology and eschatology that are based on or allude to the GIT as well as references to Gödel in novels:

> The philosophers at the great universities were, without exception, failed mathematicians. When they were not examining much of the vocabulary of civilized discourse to conclude that it, after all, lacked meaning, they muttered Gödel, Russell, Hilbert, liking to imply that they themselves had chosen philosophy over mathematics to give themselves a wider, though related intellectual field (Renata Adler, *Speedboat*).

One can find references to Gödel in literary criticism where, according to Simon Blackburn's review of Umberto Eco's *Kant and the Platypus*, "prominent literary intellectuals often like to make familiar reference to the technical terminology of mathematical logic."

One such person is reported as opining that "Gödel showed that every theory is inconsistent unless it is supported from the outside. Derrida showed that there is no outside" *(New Republic Magazine*, Feb 7, 2000).

There are web chats galore on the single topic of "False Applications of the GIT," although what is a "false" application of mathematics and what is a "true" application defy formalization, let alone common agreement.

SOME "APPLICATIONS" OF THE GIT

I use the word "application" here to mean simply that an argument of some sort has been put forward based in some way on the GIT. The GIT serves as a point *d'appui* for both specialists and the laity.

GIT suggests there is no final theory of physics (Stephen Hawking).

GIT suggests that physics—identified with mathematical physics—might be inconsistent.

GIT suggeststhat not everything that is technically desirable is technically possible (Jerome Wiesner).

GIT suggests that "whether we admit it or not all (political, social, military) actions end in the logic of triage (i.e., judgements of priorities of action)" (Hans Magnus Enzensberger, *Civil Wars,* 1994).

GIT suggests humans are not computers. Creativity and intuitive powers are not the product of computer programs (Roger Penrose).

(This last position has been seriously questioned by Murray Gell-Mann.)

"I would be skeptical about the use of the GIT (as in Penrose, 1991) for arguing the limitations of any kind of intelligence" (Steve Smale).

GIT suggests that mathematics will become more and more experimental (Chaitin).

GIT suggests that the foundations of mathematics both philosophically and technically must come to grips with stochasticity (i.e., the probabilistic element) (Gregory Chaitin, David Mumford).

GIT suggests that there may be a "high level way of viewing the mind/brain, involving concepts that do not appear on lower level" (Donald Hofstadter).

GIT suggests that mystical experiences may be the only road to absolute knowledge (Paul Davies).

GIT suggests that since "the consistency of mathematical systems becomes an incalculable question. Thus, even the exercise of mathematics involves an act of faith" (John Polkinghorne, Physicist and Anglican Priest, *One World*).

From these last two quotes, it is an easy step to say the GIT suggests that God exists.

GIT suggests that "a religion based on a plurality of religions may leave us forever struggling with an Axiom of Religious Choice" (Sarah Voss, mathematician and minister, *What Number is God?*).

In a totally different direction, GIT may give aid and comfort to the creators of computer viruses:

"For most plausible definition of 'virus', it is likely that the GIT blocks the possibility of writing a program that accepts all non-viruses and rejects all viruses" (Ernest Davis).

Thus, while each type of virus may be overcome on an individual basis, no panacea can be found. If the use of the word "virus" in both medical and computer contexts is more than mere verbal play, the GIT may suggest that a universal medical cure may be an impossibility.

Finally, there are features or "applications" of GIT that feed into psychology. The nicest, most amusing exposition of this occurs in Apostolos Doxiadis' novel referred to above..

THE PAST AND PRESENT INDIFFERENCE OF THE MATHEMATICIANS TO THE GIT

If the GIT has caused turbulence in philosophy, if it has caused earthquakes in logic, if discussions of the GIT clog the printed page and the websites, how can it be said that GIT is of little significance? And yet, despite the storms that have raged and opinions altered, research mathematicians—with the possible exception of a few number theoreticians—have little regard and less use for the GIT.

One colleague put it this way:

"I've been to many mathematical lectures and scientific meetings all over the world. Not once did the name of Gödel come up."

Another colleague told me:

"I've never lost any sleep over the GIT. But I'm sure that Hilbert did."

Therein lies the paradox.

More generally, of course, research mathematicians have had little use for mathematical logic or for the philosophy or history of mathematics. None of these is required knowledge for a Ph.D. in mathematics (nor hardly even for computer science). The idea, for example, that mathematics proceeds rigorously and rigidly from assumptions to conclusions by a set of allowed logical steps, simply does not correspond to the way that mathematics is either discovered, developed, accepted, justified, applied or presented.

If someone points out that, in principle, all accepted mathematical proofs can be written out in the manner of Russell and Whitehead's proof that $1 + 1 = 2$, I would say the phrase "in principle" is one of the weaseliest expressions in the vocabulary of intellectuals. In principle, a contemporary Robinson Crusoe thrown naked onto an island well supplied with all raw materials could produce an automobile that worked.

Mathematicians work using traditional materials and guidelines and have their own criteria for acceptance. A real conceptual or metaphysical breakthrough occurs perhaps every fifty years. Afterwards, the logicians and philosophers move in and tell the world what, exactly, the mathematicians have been doing.

Mark Steiner comments on the sociological phenomenon of mathematicians who ignore logic completely in their description of the history and philosophy of 20th century mathematics and yet cite the GIT as one of the most important recent results. His recently appeared *The Applicability of Mathematics as a Philosophical Problem* does not mention the GIT.

While the GIT is a piece of mathematics created along traditional lines, but applied to mathematics self-referentially, it must be regarded as an "inside job." On the other hand, since it seems to limit what mathematicians will ever be able to accomplish, limiting the independence of mathematical action, it is also an "outside job."

Numerous people have walked away from certain parts of mathematics, either for conceptual or for utilitarian reasons, e.g., measure theory. They do not accept it. It has no gut meaning for them nor relevance to their scientific work. The GIT paradox is part of this phenomenon.

THE GIT AND THE FAMOUS UNSOLVED (OR ONLY LATELY SOLVED) PROBLEMS

In Doxiadis' novel, his hero uses GIT as an excuse for calling quits to his intense labors on the Goldbach Conjecture.

The mathematical world is full of unsolved problems and conjectures. Most conjectures fail to gain notoriety, primarily because they are not associated with a "great name." Mathematicians lose interest in them, so hence they are not worked over for long periods of time. Many of these, in number theory especially, have been listed in such books as Daniel Shanks' *Solved and Unsolved Problems in Number Theory* and Richard Guy's *Unsolved Problems in Number Theory.*

> **"*One might very well call such a piece of mathematics a fundamental symbol or myth...***

Neither of these books breathes the name of Gödel. It is probably the case that most of the conjectures listed in these books are decidable one way or another. The difficulty or the depth of a conjecture can only be guessed, but some measure of it may be gleaned from the rewards ($25, $100, etc. 100,000 pre-WWI German marks for Fermat) that are sometimes offered for a solution by some of the proposers.

THE GIT: ONE OF THE FUNDAMENTAL MYTHS OR ARCHETYPES OF MATHEMATICS?

The fact that the GIT has contemporary applications, implications or suggestions relative to a wide variety of fields ranging from cognition, physics, and philosophy, to literature, theology, and politics, gives it a special and remarkable status among mathematical statements. The educated laity seem to be attracted to it as iron to a magnet or as the devout to an icon. The name Gödel can create a best seller or fill a large lecture room. It can also do the reverse. This is part of the paradox and adds to the unique status of the GIT. It would be impossible to make such wide claims for, e.g., Gershgorin's theorem in matrix theory, or indeed for any of the theorems employed routinely in daily research. One would have to go back to the mental world of the Pythagoreans or neo-Platonists (ancient or contemporary) to find statements, contexts and attitudes of equal popularity.

One might very well call such a piece of mathematics a fundamental symbol or myth in the sense of the psychologist Jung. Jungian archetypes carry many interpretations; it is also the case that many explanations have been advanced for the Paradox of Gödel.

A BASKET OF EXPLANATIONS OR DENIALS

My object now is to record a wide variety of reasons that have been given to explain the Paradox of Gödel and then to set forth my own naive reasons.

> The GIT is equivalent to Turing's theorem about the unsolvability of the Halting Problem. I don't think that has any practical consequences for real life computation which deals with finite memory and finite computation times. It just shows the vast gap between what is of metaphysical interest and practical interest. Similar discrepancies abound in economics and in fluid dynamics (Reuben Hersh).

Tying the matter a bit more closely to the philosophy of mathematics, Hersh goes on to say:

> It seems to me that most or all issues of mathematical philosophy are important in some sense independent of concrete specific examples or applications.
>
> For instance, is there really an infinite set, or is it just something we imagine? From a philosophical viewpoint, this is a very basic, fundamental question. But for mathematical work, it doesn't make any difference, and many mathematicians couldn't care less about it.

> Gödel's Incompleteness Theorem is a mathematical result with philosophical import. It has limited mathematical import. Which shows that mathematical and philosophical import are not the same thing (Hersh).

> For mathematicians, however, his [Gödel's] theorem was of marginal interest, since Gödel worked with a far more formal definition of proof than that to which they aspired (or still do); so the separation of logic and mathematics continued largely unchanged (Ivor Grattan-Guinness, *The Rainbow of Mathematics*).

> I don't find it [the paradox] paradoxical. You can compare the GIT to Liouville's proof of the existence of transcendental numbers. It is an example of a phenomenon, but it is of no help for interesting number like e or π...One aspect of the matter, not a direct consequence of the GIT, but coming out of that development, is the work on unsolvable decision problems that has had a serious impact in certain fields, e.g., finitely presented groups (Martin Davis).

"The infinite is not the issue. It is the case that the GIT has implications for finite memory and finite computation time" (Ernest Davis).

> As far as applied mathematics goes, there is considerable evidence that all scientifically applicable mathematics depends on weak systems of set theory, even conservative over arithmetic. No new axioms are necessary (Sol Feferman).

Question: does Feferman's observation about weak systems constitute a descriptive hypothesis that limits the structure of physical theories just as the Church-Turing Hypothesis (the Turing machine models all possible computations) limits the nature of computation?

> Most mathematicians don't know or care about logic and they see the GIT as a kind of curiosity. It says nothing about the undecidability of the problems they happen to be working on. It provides no decision procedure for deciding beforehand whether a given statement in mathematics is or isn't decidable.

> If they took the possibility of undecidability seriously, if they agreed, for example, that with probability one, a proposition given at random is undecidable, they would be discouraged away from the field. Mathematicians use their insights, judgements, experience, to enable them to focus on statements which turn out to be decidable(John Casti).

> A parallel from physics The response of mathematicians to GIT has been rather like the response of physicists to general relativity in the period roughly from 1916-1960. Physicists understood that Einstein's results were in some way quite fundamental, but because general relativity seemed so definitely a singular achievement, physicists tended to ignore its implications while ceremoniously paying lip service to its grandeur...

> Mathematicians are instinctively inclined to assume that if the GIT and nearby results are as important as logicians seem to think they are, then it should be possible to use those results to discover something beyond the results themselves. Nothing has yet emerged.

> It is possible for a result to have immense importance for a discipline without leading to anything interesting within the discipline (David Berlinski).

It is not the case that GIT has contributed little to math or computer science. A fair number of interesting problems have been proven unsolvable using a reduction to the GIT. The best known is Hilbert's Tenth Problem. There are numerous other results in number theory, logic, computation theory, discrete math and algebra, e.g., the Paris-Harrigton result which states that the Ramsey theorem is not provable within number theory.

But just wait a bit! Things might change! Mathematicians have tended to ignore the GIT because it seemed to have no connection to other parts of mathematics. However, in the past twenty years, this has changed somewhat. Harvey Friedman's work has shown that incompleteness theorems do have a very real meaning for number theory. But the fact remains that the connection is weak in that it seems to point to nothing more than oddities in the structure of arithmetic. This may or may not change.

ANTI-GÖDELIAN DOUBTS

The GIT seems to have come as a surprise to neither to John von Neumann nor to Norbert Wiener (S.J. Heims: "John von Neumann and Norbert Wiener").

"The mathematical fraternities' actual experiences with its subject give little support to the assumption of the existence of an a priori concept of mathematical rigor" (John von Neumann, *The Mathematician*).

Further down the spectrum there are anti-Gödelian doubts:

"...it is commonplace that Wittgenstein rejected Gödel's proof [i.e., the GIT] because he did not, or even could not, understand it" (Juliet Floyd).

> The GIT is based on a chimera. The formalization of mathematics assumes its representation in a set of recognizable signs that are beyond questioning. The metamathematics however is stymied by the ambiguity (incoherence) in how those signs are actually viewed. The metamathematical argument of the GIT collapses into confusion. Since the whole enterprise of formalization is not feasible, GIT is redundant. No wonder mathematicians are not bothered by it in their work (Miriam Yevick).

THE WAY I SAW THE PARADOX

I first heard of the GIT around 1941, when I was a college undergraduate. The GIT was then ten years old. It caused no alarm in me. The bottom line seemed quite reasonable. There were mathematical problems I could not solve. I had heard that there were problems that no one had yet been able to solve. I knew that there were problems, which, as stated, provably had no solution.

An example: working in the plane, connect three houses by curves, to the "electric, gas, and water works" so that the curves do not intersect. (But in real life we connect them in 3-d).

Another example: the squaring of the circle by ruler and compass. Or, perhaps more significantly, the "demonstration that Euclid's Parallel Postulate" cannot be derived from the other postulates. Thus, there appear to be many problems that were impossible to solve in the way they have been formulated.

This being the case, and arguing by analogy, GIT seemed to me to be reasonable. As in the Fifteen Puzzle cited in the Epigraph, you might not be able to get "there" from "here." Of course, these examples are specific problems within mathematics and the GIT is a theorem about theorems. Up a metalevel, or is it?. But the proof of the independence of the parallel axiom is a proof that there can be no proofs of dependence. So the disparity of levels did not bother me. However regarded, these analogies were strong enough for me. (But not strong enough, apparently, for Frege, Russell, Hilbert, *et al.* Is this yet another paradox?)

The idea of mapping formulas onto integers (Gödel Numbering) seemed ingenious, but a bit dubious. The Gödel numbers are so large! What kind of existence can be attributed to them ? Do they really function in the way that 1, 2, 3 do? Are these numbers being used in different ways that really do not mesh with one another or with the numbers of everyday arithmetic? (I was, and still, am a "weak finitist.")

And then came the *coup de grace*. Nothing but a complicated form of the Liar Paradox. Hence a self-referential swindle, a trick of language.

So while I was quite willing to accept the bottom line of the GIT, I did not care much for the proof. I did not need the whole Gödelian apparatus to convince myself that I couldn't lift myself up by my own bootstraps either physically, mentally, or mathematically.

To add to my undergraduate skepticism, why was the world famous logician W.V.O. Quine, with whom I was even then studying mathematical logic, in the Department of Philosophy at Harvard and not in the Department of Mathematics? Obviously, the Harvard Mathematics Department considered mathematical logic to be irrelevant to their interests. In point of fact, this was my first perception of the Paradox of Gödel.

MY CURRENT VIEWS

To discuss GIT and the Paradox, as Reuben Hersh pointed out above, one might very well go into the logic, the philosophy and metaphysics of mathematics, metamathematics and cognition. What is a legitimate mathematical object, existentially? What are legitimate constructions or operations? What is truth? What is proof? How can we recognize what makes sense and what doesn't? What does it mean to "know"? What sense does it make to say that "it will never be known whether the statement X is true or false"? What does it mean to explain anything?

I shall bypass all these. I will not look for an explanation in terms of logical structures or the relative strengths and weaknesses of axiom systems. I will go for what might be called a historical view of the matter.

Toward this end, one should realize that at various times actual mathematical practice has been other than what it is claimed to be in an ideal and hence limited sense. Over the years, I came to believe that the "standard" view of mathematics as consisting of hypothetical-deductive structures is a totally inadequate description of how I (personally) have understood and internalized mathematics; how I applied mathematics to itself or to the outside world, or how I created new mathematics.

Historically, there are many times and places in mathematics where mathematics has said "impossible," "no way." Some of these impossibilities are hinted at in the persistence of old mathematical terminology, e.g., negative, irrational, imaginary numbers. Another impossibility: no general formula involving a finite number of simple operators and root extractions can be found for the solution of the quintic equation. Yet, the history of mathematics displays all these and many, many more impossibilities and contradictions (e.g., Heaviside's operational calculus; Dirac's delta function) being bypassed, legitimized, co-opted, often by the method of context extension.

I began to wonder about the notion of proof, a process absolutely fundamental within a certain view of

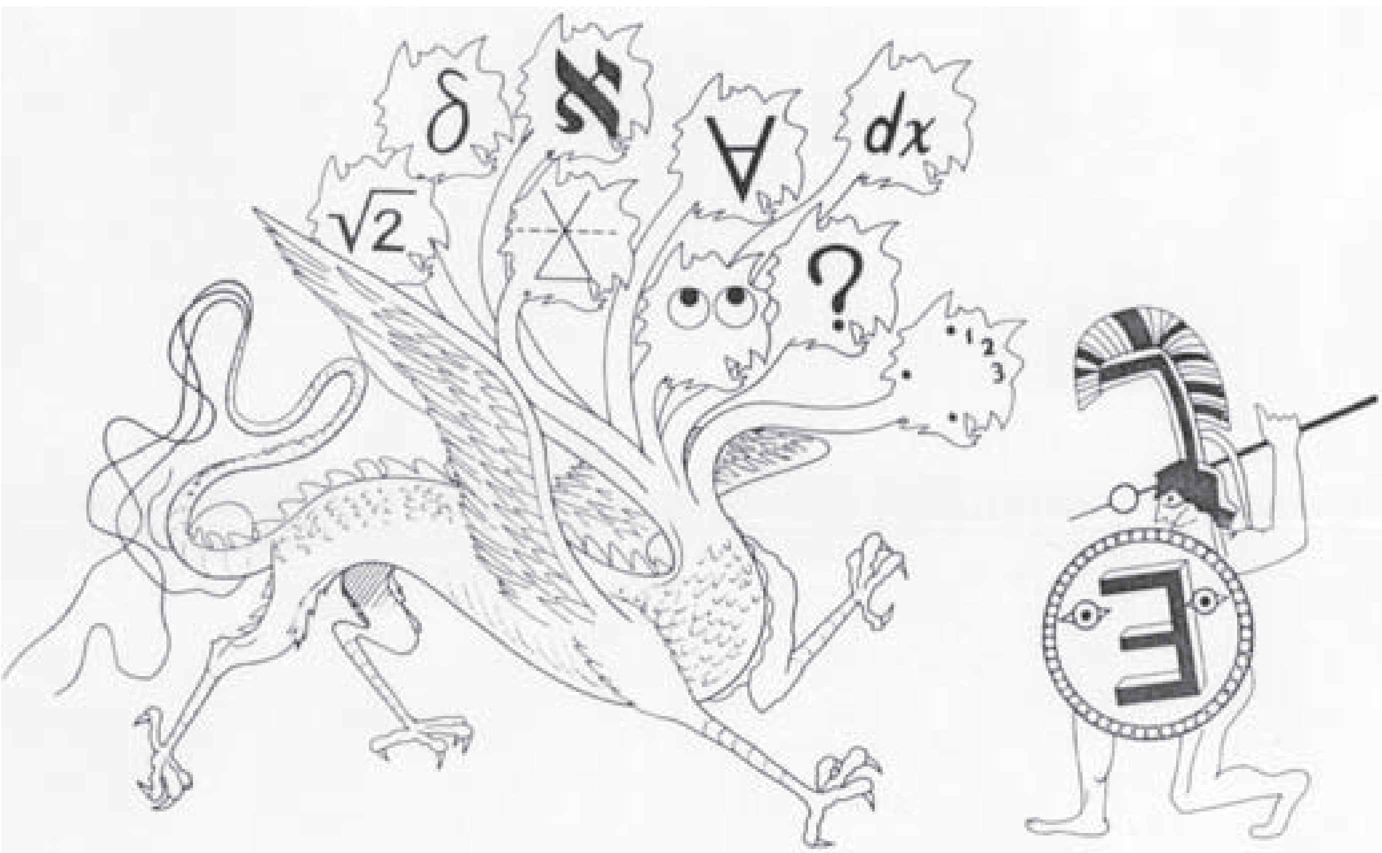

Figure 1
The Hydra of Mathematical Impossibility is slain by the Hercules of context extension. (From Davis and Park, 1987.)

mathematics, but proof was not identical understanding. Moreover, proof was subsequent to deciding that initially there was something there to prove. And the axioms were statements designed post hoc long after a substantial corpus of mathematics was in place. (In the case of arithmetic with Frege in 1884! Was there no valid arithmetic till then?).I began to feel that what was important was "mathematical evidence,"of which proof is only one component.

I began to wonder about the concept known as consistency. To be inconsistent, mathematically speaking, is to commit the primal sin. If one allows one contradiction, one can demonstrate anything at all. But can one really say with absolute objectivity, finality, and without relativistic allusions, what consistency consists of when there is a historical record of a constant patching up of mathematical inconsistencies in a way that makes them disappear? (Imre Lakatos)

An up-to-date example: consider the arithmetic system that is embodied in the popular and useful scientific computer package known as MATLAB. MATLAB yields the following two contradictory statements:

(The symbol == means "is the equality true or false?")

	Input	Output
(1)	l e-50 == 0,	false
(2)	2 + l e-50 == 2,	true.

Yet, MATLAB arithmetic is a (finite, but large) mathematical structure. Operations can be carried out. Certain inputs lead to certain outputs. These might be called MATLAB truths or theorems. The computation itself is the proof or the validation of these truths. They are deemed useful by the scientific community. The structure has its own integrity in that it consists of just what it consists of and it does just what it does. Yet, when judged by certain other ideal structures MATLAB embodies contradictions. While the God of Consistency does not thunder nor shake the earth in the presence of these logical irrationalities, one might well ask whether these contradictions can lead to error or disaster when MATLAB is employed in physical applications. They can, but "knowledgeable" programming makes the likelihood small. In any case, "ideal" mathematical computations (if indeed they can be carried out) might also lead to disaster.

Incidentally, I believe that Wittgenstein deplored "the superstitious fear of mathematicians for contradictions" (quoted in Karl Menger's *Reminiscences of the Vienna Circle*).

Rejecting mathematical platonism, formalism, logicism, and constructivism, I adopted a position that has been called variously "social constructivism," "quasi-empiricism," or "humanism."

THE PARADOX OF GÖDEL: MY EXPLANATION

Is this an explanation? Not really; just some thoughts conjured up by thinking about the paradox.

Mathematics is a living organism. The modes of its discoveries, developments, justifications, and interpretations cannot be formalized in a few paragraphs– if at all. They are time dependent and hence cannot be set down once and for all.

The development of mathematics either as a manufactured or a discovered corpus, goes forward to a great extent without set global goals. As it goes forward, year by year, what it turns up can be quite fortuitous, serendipitous, perhaps even interesting; in such a case, the arrival at a theorem is automatically accompanied by evidence of its validity or relevance, sometimes even by its proof.

Mathematics moves forward from statements already in place that suggest other statements. One of the goals then becomes to arrive at a proof of the suggestions. But the researcher is borne forward by a trust in a kind of "principle of continuity" (which admittedly can be dead wrong, see the "Fifteen Puzzle") implying that statements "close" to proved statements are themselves provable or disprovable.

In the older Eastern tradition, explicit proof is often missing. In the Western tradition, the notions of what is proof and what is provability evolved slowly and simultaneously with the discovery or creation of much material that was in fact provable. Alongside this, there grew the dominating or establishment view of mathematics as a logically deductive enterprise. The steady supply of proofs and the demand for more, interacting upon one another, grew together. The characterizing notion of mathematics as proved theorems grew with equal steps with the success in proving those theorems even as the notion of what constituted a valid proof altered and changed with time. The concept of what constitutes a proof has no finality; it develops alongside the material on which it operates.

Contemporary histories of mathematics present what is often called "Whiggian history." That is, they promote the current established view of mathematics as deductive structures, and they interpret the mathematical past as leading inevitably to the established present.

If it should have turned out that a good many of the statements deemed interesting by mathematicians or scientists were unprovable, or undeterminable whether they were unprovable, then the view of the mathematical enterprise as it developed in the 19th and 20th centuries, an enterprise that set increasing store by deductive proof, would have become untenable.

> "
> ***The concept of what constitutes a proof has no finality; it develops alongside the material on which it operates.***

In such a case, mathematics would not have disappeared. It would be an art with a special vocabulary and modus operandi, a form of rhetorical discussion, a set of procedures, suggestions or rules as to how the world might be organized, and the GIT would be both true and irrelevant.

AS REGARDS THE FUTURE.
In my opinion the most significant mathematical development of the 20th century has been the computer in all its ramifications, mathematical, scientific and social. Having done scientific computation in the preelectronic days as well as with contemporary very-high-level "tool kits," I still tend to think of the computer as a "mathematical instrument." But this view and a related view that the computer is a "logical engine," an "algorithm cruncher," though historically accurate, may now be as obsolete as the horse and buggy. What is replacing it?

Programmed computation, i.e., algorithms, and deductive proof have common features. But the future dominance of the algorithm–and the GIT is algorithmic in structure–has been questioned. Since it is fundamental to all digital communication in the same way the elementary particles of physics are fundamental to a Hawaiian wedding luau, there are some signs that the algorithm may have to share the center stage of technological and instructional emphasis or even to retire to the wings.

Here are a few straws in the wind:

Computer scientist Peter Wegner thinks that in the future the emphasis will shift from algorithmic models of computation to interactive models. We have now reached the point where a single computer has become a basic "elementary particle" of information interaction, to be combined with myriads of other individual computers and acted on non-algorithmically by the whole exterior environment, human and non-human.

"Interactive systems are grounded in an external reality both more demanding and richer than the rule based world of noninteractive algorithms" (Wegner).

"The conventional metaphor [for computation] will be replaced by the notion of a community of interacting entities" (Lynn A. Stein).

By way of a parallel within mathematics:

> Mathematics has been regarded traditionally as 'theorems.' It is now becoming the study of structures. Until the 20th century, there have been only two structures: geometry and arithmetic. Now there are many (David Mumford).

(MATLAB , mentioned earlier, is just one of the more fairly recent ones.)

While by no means neglecting the algorithm, we must surely add to the idea of structures the notion of stochasticity as a prime element of the future composition of mathematics.

"The intellectual world as a whole will come to view logic as a beautiful elegant idealization but to view statistics as the standard way in which we reason and think" (David Mumford).

Working with the material of a structure and employing its rules, the mathematical culture goes forward from "here" by successive steps and arrives at a "there." Often probabalistically and non-algorithmically and even multivalently. The delivery of the "there" from the "here" may be regarded as proof in an extended sense. Whether the "there" is in any way interesting or appealing or suggestive or useful or whether it corresponds to a "there" desired in advance is altogether another issue.

> By now the ideas elaborated by Gödel, Church, Turing, and Post have passed entirely into the body of mathematics where themes and dreams and definitions are all immured, but the essential idea of an algorithm blazes forth from any digital computer, the unfolding of genius having passed inexorably from Gödel's Incompleteness Theorem to Space Invaders VTT rattling on an arcade Atari (David Berlinski: *The Advent of the Algorithm: The Idea that Rules the World*).

The GIT is moving off center stage, a place it never really occupied. Like the ideas of Freud which now appear more in literature than in therapy, it will survive as an archetypal statement from which all kinds of inferences–mainly non-mathematical–will continue to be drawn.

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"The development of mathematics towards greater precision has led, as is well known, to the formalization of large tracts of it, so that one can prove any theorem using nothing but a few mechanical rules."

--Kurt Gödel

Problems in Which Given Information is Ignored

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SUMMARY
A set of problems is presented and discussed for which there is a tendency for students to ignore part of the given information in the problem and to substitute some extraneous assumptions. Typical student reactions are also discussed.

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This article is about a very interesting and very specific class of non-routine mathematics problems. I discovered this particular type of problem in part from teaching a mathematics content course for prospective teachers that has a strong emphasis on problem solving, and from giving workshops to prepare prospective teachers for the mathematics questions on the New York State Teachers Certification Examination.

Of course, there have been many books and papers written about problem solving in mathematics (for example, [2], [5]). There have been studies of problems with too little information, problems with redundant information, problems with no possible solution (for example, [1]), and many other classifications. However, the class of problems to be discussed here does not seem to have been considered separately.

Here is a simple initial problem to illustrate the main idea. Krutetskii ([1], p. 142) gives the following problem:

One leg of a right triangle is equal to 7 cm. Determine the other two sides if they are expressed in integers.

The solution would be that the other two sides are 24 cm and 25 cm, but Krutetskii states that students faced with this problem would often claim that it could not be solved because a triangle cannot be determined if only one side is given. The student is forgetting, or perhaps ignoring, the condition that the other two sides must be integers. This raises the question of what causes the student to ignore this condition. Is it that (s)he does not know how to use this information? Or is it that (s)he does not think that the condition is relevant?

This is an example of the class of problem I wish to consider here, namely problems *in which the student tends to ignore some given condition in the problem*. I have found remarkably few problems which seem to fit this requirement exactly, and I will present them here. It is interesting to consider just what it is about these particular problems that causes the student to ignore a relevant portion of the given information in a problem. Frequently, the students are inclined to substitute some extraneous assumption of their own for the information which they are ignoring, as will be illustrated in the following examples.

The problem which originally suggested to me this idea of ignoring part of the given information is the following ([3]):

Four men, one of whom has committed a crime, made the following statements:
Arch says: Dave did it.
Dave says: Tony did it.
Gus says: I didn't do it.
Tony says: Dave lied when he said I did it.
If only one of these statements is true, who was the guilty man?

I have presented this problem at large workshops, and I have been somewhat surprised to find that invariably the students ignored the condition that only one of the statements is true. Typically they suggest reasons for their choice that are based on some sort of psychological argument, for example: "Gus must have done it, because he didn't accuse anyone else," or "Tony must have done it, since he accused Dave of lying." The students are substituting some sort of extraneous assumption for the given condition that only one statement is true, perhaps because they do not see how to apply this condition, or do not see how it is relevant. To solve this correctly, one could use this condition to test the possibility of each man's guilt. This would reveal that the correct answer is that Gus was guilty, for only in that case is exactly one statement true; if any of the others were guilty, then two or more of the statements would be true. Perhaps the idea of considering separate cases (i.e., what if Arch is guilty, what if Dave is guilty, etc.) does not occur to the students because they have not experienced many problems which are solved through the strategy of using cases.

Another example of this phenomenon appears in [4]:

Exactly one of the following statements is false:
a) Audrey is older than Beatrice.
b) Clement is younger than Beatrice.
c) The sum of the ages of Beatrice and Clement is twice the age of Audrey.
d) Clement is older than Audrey.
Who is the youngest—Audrey, Beatrice, or Clement?

Here too, students are inclined to ignore the condition that exactly one of the statements is false. For example, they may look at only two of the statements, say (a) and (b), and conclude from them that Clement is the youngest, ignoring the presence of the other statements. Similarly, they may look at only statements (a) and (d), and conclude quickly that Beatrice is the youngest. Occasionally they will see that statements (a), (b), and (d) contradict each other and will erroneously conclude that the problem cannot be solved. The students may tend to pay less attention to statement (c) because it is harder to understand, or because it involves consideration of the arithmetic operation of summing. They fail to realize that statement (c) implies that Audrey's age must be in between the other two ages (if all ages are different). Again, one could simply consider each of the four statements in turn, and ask, "If this statement were the false one, could the other three be all true simultaneously without contradiction?" In this way one could eventually determine that the only possibility is for (b) to be the false statement, which would make Beatrice the youngest.

A fourth problem, which the readers may have seen in one form or another, is the following:

Mrs. Adams and Mrs. Brown, two math teachers, are walking over to Mrs. Brown's house after school.
Mrs. A: How many children do you have?
Mrs. B: I have three children.
A: What are their ages?
B: The product of their ages is 36.
A: [Thinks for a moment] That's not enough information to figure out their ages.
[By now, the two of them are at Mrs. Brown's driveway, so that Mrs. Adams can see the number on Mrs. Brown's house.]
B: The sum of their ages is the number on the house.
A: [Thinks for a moment] That's still not enough information, I still can't figure out their ages.
B: The oldest child is visiting her grandmother.
A: [Instantly] Now I know their ages!
What are the ages of Mrs. Brown's children?

To solve this problem, one would first list all combinations of three whole numbers whose product is 36, as possible candidates for the ages of the three children. When Mrs. Adams is told that the sum of the ages of the children is the number on Mrs. Brown's house (which she knows), she states that she still cannot determine the children's ages. Among the triplets of whole numbers whose product is 36, only two such triplets have the same sum: 2+2+9=13 and 1+6+6=13; all other sums are distinct. Therefore if the house number was anything other than 13, Mrs. Adams would know the ages of the children as soon as she is told that the sum of the ages is the house number. The fact that she still cannot determine their ages at this point implies that the house number must have been 13. Then, when Mrs. Adams is told something about the "oldest child," she knows that the answer must be 2, 2, and 9, because in the combination of 1, 6, and 6, there is no "oldest" child.

In this problem almost invariably students will ignore the given fact that the sum of the children's ages is the number on the house. It seems apparent that this is because they do not see how this information can be used without the actual value of the house number. In fact, when the problem is posed, the students often ask, "What is the number on the house?" Sometimes they will state that the problem cannot be solved without the house number being given. In this problem I have seen all sorts of extraneous assumptions introduced to replace the clause which is ignored. For example, they may assume that the oldest child must be at least, say, 12 years old in order to be allowed to visit her grandmother on her own. Or, they may assume that the oldest child must be under 6 years of age (i.e. the answer must be 3, 3, and 4) because otherwise she would have to be at school that day (it is a school day, since the teachers were going home from school!) and could not be visiting her grandmother. In one highly unusual case, a student came to me and asked "Is the grandmother dead?" The student explained that she wanted to know this because if the answer was yes, then the oldest child was actually visiting her grandmother's grave site at a cemetery, and she thought that children under age 18 might not be permitted to visit a cemetery. It seems that students think that the grandmother is somehow relevant because they are accustomed to textbook problems in which only the necessary information is given, and they assume that if the grandmother were not relevant, then she would not be mentioned.

The four problems shown above have in common a tendency for students to ignore an actual given condition in the problem. Below is one more problem which is closely related in this regard, but with a slight difference:

What is the greatest amount of money (i.e., the maximum VALUE) in coins (up through half-dollars; no dollar coins) that you can have and still not be able to give someone change for any of the following: a nickel, a dime, a quarter, a half dollar, or a dollar?

In this problem students sometimes erroneously assume that you must have at least one of each type of coin (from a penny through a half-dollar, inclusive). Alternatively, students may incorrectly assume that the amount of money must be less than a dollar. (Actually the correct answer is $1.19; a half dollar, a quarter, four dimes and four pennies, but no nickels.) This is similar to some of the preceding problems in that students tend to make extraneous assumptions. However, it differs from the foregoing in that in this case the assumptions which cause the students to overlook *possible solutions* do not involve ignoring a given *condition* of the problem, as in the earlier examples. However, the tendency to overlook possible solutions is more common than altogether disregarding a part of the given information in a problem.

An interesting but perhaps difficult question for future study would be to examine why it is that students respond differently to these types of problems as compared with other non-routine problems. Is it possible to identify some commonality among these problems which provokes this unusual response, and how can we help our students to focus more on the meaning of the given information rather than introducing superfluous assumptions? Finally, I would be interested in hearing from readers any suggestions of other non-routine problems which would fall into the category discussed here.

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"Common sense is the collection of prejudices acquired by age eighteen."

--Albert Einstein

The Need to Diversify the Ranks of Teachers of Mathematics

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ABSTRACT
This paper intends to give rationales for the need to diversify the ranks of teachers of mathematics. It also suggests ideas that we can take to alleviate the situation that we currently are confronting.

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"It is essential to have a workforce of strong mathematics teachers that reflect the demographic characteristics of the student population." This claim is taken from NCTM's position statement on *The Mathematics Education of Underrepresented Groups*. My personal feelings dictate that a reader's immediate idea of characteristic is purely based on ethnicity. However, the characteristics (I prefer referring to them as *gaps*) of our classrooms encompass many other points that, to name a few, include gender, culture, language, socioeconomic status, physical and learning abilities, etc. No one can argue against the fact that our classrooms today are getting more and more diversified. Keeping up with these changes is indeed not only challenging but also important. Allow me to discuss some of these gaps.

One of the known gaps we have right now is gender. According to the report of the National Education Association (NEA) in 1997 entitled "Status of the American Public School Teachers, 1995-96," there were about 2,164,000 teachers in the USA. Of these teachers, females comprised 74.4% as compared to 25.6% of male teachers! Aside from the incongruity in the ratio of female to male, it also is necessary to look where teachers are primarily concentrated. Most of the female teachers are in elementary schools, and most of the male teachers are in mathematics and science disciplines and most likely in high schools. The way I look at it, this is a troubling scenario since our students now have fewer and fewer role models to look up to. Let's face it, some issues and incidents that students confront are better confided to someone of the same gender.

Next comes the racial makeup of our teachers and students. If one asks what percent of K-12 students are minorities, would you know it to be nearly 30%? Focusing on the teachers, do we know it to be about 13%? Or that over 40% of schools in the US do not have a faculty member who is a person of color? Quite eye-openers, aren't they? We can no longer deny the fact that our students are getting more and more diverse, but our teachers do not match the rate. Is there really a reason for concern? Frankly, yes, there is. A report of The Mathematical Association of America (MAA) entitled *Attracting Minorities into Teaching Mathematics Executive Summary* provided one good reason by stating that "a diverse teaching force shows all students that minorities can do mathematics and that diversity is a positive component of American society." Another reason is that we need to show these students that they too can achieve and that mathematics is intrinsic to us. Our students must be set to succeed in their undertakings. We, as teachers, should lead our students to the right avenues.

Reading the NCTM position statement again, the word strong is an area of concern. I do believe that when we say "strong" we mean "qualified." With what we currently have, our teaching force is not "strong." According to the *Digest of Education Statistics 1997*, of a total of 1,158,788 Bachelor's degrees awarded in 1994-95, 9% were in Education and 1.2% were in Mathematics! Of the 397,052 Master's degrees, 25% went to Education and 1% went to Mathematics. Furthermore, more than one-fourth of newly hired teachers do not have the necessary license. To complicate the situation, 12% have no license at all, and another 15% have temporary licenses. Add to that the fact that 22% of our new teachers abandon the profession within the first three years. Do we even dare to look into how many teachers leave after ten years?

What can we do? We need to make these situations known to the public. We no longer can be silent and wait for things to happen; otherwise things will just bypass us. We need to speak to hiring bodies about the necessity to diversify the ranks of mathematics teachers. We, as a world, are getting closer each time, and each day that passes brings this reality more and more real. Many actions still are needed to achieve equity and to have an effective teaching work force. More recruitment is needed, and the profession has to be made more lucrative and more inviting. We need to write to our lawmakers to demand that this predicament be looked at and to lobby for the improvement of teaching conditions. Actions need to be taken, and they have to be taken now! Time, as we all know it, is fast running out. It will be sad for us to realize that because of negligence, this problem will be beyond rectifying.

What I have presented here are current realities in our classrooms. If we are to uphold GOALS 2000 in making sure that there *will be a talented, dedicated, and well-prepared teacher in every classroom*, we have to consider and make this a priority! Let us be reminded that in the next ten years we will need at least two million new teachers. If things remain status quo, the question is no longer, "What kind of future will we have?" but simply, "What future?"

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Educating the Public about School Mathematics

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As I polish these remarks, it is 11/5/99. Need we say more to realize that our calendar is a mathematical model of time? This model is based on our position in the universe. One orbit of the Sun is a year. We judge our age in orbits; we often think of both current events and history in terms of tens and hundreds of orbits—that is, in decades and centuries. This shows the influence of base 10 on our thinking. As we hit the juncture of the beginning of an orbit numbered 2000, we are reminded of this mathematical model.

This seems to be an appropriate time to review recent orbits. My goal is to do this in a way that will be interesting and informative. I have picked the last 50 orbits as my time frame because this interval covers the schooling of most of your students and their parents.

THE NEW MATH ERA

There is also a conceptual reason for beginning in 1950. In 1951, three faculty members in mathematics and education at the University of Illinois began the first of the new math projects, UICSM. Six years later, in 1957, the new math received its biggest push when the Soviets launched Sputnik, the first artificial satellite. Sputnik was neither a small nor an isolated feat. It built on the work of German rocket scientists that had started 15 years earlier in World War II and its 186 pound weight, followed the next month by the half-ton Sputnik II, showed that the Soviets had the capability to send a large missile anywhere in the world.

Within a year, the U.S. Congress passed the National Defense Education Act, which included sizeable funds for curriculum reform. These funds allowed the fledgling School Mathematics Study Group (SMSG), which had been initiated just a year before, to become the largest research and development project in mathematics education the U.S. has ever seen.

The work of SMSG was hailed widely by all connected with mathematics and education. The euphoria of the time is perhaps best represented by a 1963 report of The Cambridge Conference on School Mathematics entitled *Goals for School Mathematics*. In it, a group of 25 distinguished mathematicians from Harvard, MIT, Stanford, and other top universities joined mathematics educators and other professors of education in an attempt "to express their tentative views upon the shape and content of a pre-college mathematics curriculum that might be brought into being over the next few decades." [p. iii]

These mathematicians were strongly affected by the modernization of mathematics that was the trademark of UICSM, SMSG, and the other new math projects, and the successes that the projects seemed to be having. It led them to believe that students could learn much more if the mathematics were presented in an abstract, clear, and logical way. So they proposed a curriculum for grades K-6 that included conic sections, equations of lines, 3-dimensional Cartesian coordinates, polar coordinates, the vocabulary of elementary logic, graphs of relations and functions, the logarithm function and trigonometric functions. This was before the appearance of handheld calculators, but the use of desk calculators, slide rules, and tables was encouraged at these grades. In grades 7 and 8, students would study rational forms and functions, the derivative of a polynomial, the Euclidean algorithm, and a huge amount of statistics, including expectation and variance and the Poisson distribution. Two curricular organizations were proposed for grades 7-12 for the following reasons: First, as the authors wrote, "It was recognized that there are many different routes to follow in teaching geometry and that each has its advantages." [p. 47] Second, the authors believed that more than one approach to algebra and to calculus seemed reasonable, and they admitted not to know which was best. In both proposed curricula, probability and linear algebra were to be studied more than once.

Although their suggestions remain extraordinarily unrealistic, the Cambridge Conference mathematicians recognized the role they were playing: "These views are intended to serve as a basis for widespread further discussion and, above all, experimentation by mathematicians, teachers, and all others who share the responsibility for the processes and goals of American education. At this stage of their development they can not pretend to represent guidelines for school administrators or mathematics teachers, and they should not be read as such." [p. iii]

The Cambridge Conference mathematicians recognized that the difference between a mathematician and a mathematics educator is as great as that between a research biologist and a practicing physician. The physician sees patients and knows both symptoms and potential cures. The good physician realizes that not all patients are alike, and that you can prescribe things but patients don't always do what you prescribe. Mathematics teachers and those who train them and deal with curriculum day in and day out are the physicians of our profession. Teachers are the experts, and are particularly expert about their community.

In these years, mathematics educators loved the new math, and the general public liked it as well. In 1966, Francis Mueller, after studying articles about mathematics education in popular magazines from 1956 to 1965, identified those years as "happy years for 'new math"' and concluded, "As these years pass, less and less is said about mathematics being a highly disliked subject; more and more is said about the brightness of the future along these new mathematical tracts." But Mueller noted that at the beginning of 1965 there began to appear articles in *Time* and *Newsweek* questioning the ideas behind the new math. He wondered whether these articles might "mark a point of transition at which the public began to revise its perception of 'new math'." [Francis Mueller, "The Public Image of New Mathematics," *Mathematics Teacher 59* (November 1966): 621.] We know today that the public did revise its perception—completely. New math is now often treated as a debacle in mathematics education.

What is not so well-known is that the evidence for a debacle is not there. If the new math was so bad, how come the evidence is so hard to find? In fact, the evidence often leads the other way. By the early 1970s we were producing more students majoring in mathematics and majoring in science than ever before. Advanced placement programs existed in many schools where fifteen years before there was no mathematics beyond trigonometry. Enrollments were up in all mathematics courses even though many states had not changed their graduation requirements.

The public was misled by false signs of failure and a lack of sophistication about statistics that made it impossible to read these signs accurately. The first false sign of failure was a 21-point drop in SAT scores from 1963 to 1973. Although everyone should have realized immediately that something outside of mathematics was affecting performance when the verbal scores dropped 35 points in the same time period, not until 1976 did an official College Board report indicate that the drop in the 1960s was due to the much larger numbers of students taking the test.

The second false sign of failure was the appearance in 1972 of the first National Assessment of Educational Progress data on how well our 13-year-olds and 17-year-olds performed. As virtually always happens in the first administration of every large-scale test, performance was lower than people expected. But the NAEP designers were not so naïve. They also purposely tested adults who had gone to school before new math and found that the 17-year-olds outperformed those adults. This result, however, had no effect on the public view.

There was a true sign of failure. Although, overall, students seemed to be helped by new math, many students—particularly slower ones—were not well served by an abstract mathematics curriculum. These students were blown away by the new math, and their parents commiserated with them. Teachers and other adults who had been against the new math from the beginning used every instance of failure of new math students as a sign that the entire movement was a failure, and rallied public support against these curricula.

In the mid-1970s, as a response to the new math, a back-to-basics textbook series for grades K-12 appeared. It encouraged competence on skills without properties or applications, and the books contained little or no explanation. For a couple of years the elementary school texts of this series were the most purchased in the country. Though the back-to-basics high school texts were used in many places, well into the 1980s the textbooks of the Dolciani series, written in the 1960s and showing great influence of new math, remained the most used books in the country for algebra and geometry students. Honors algebra and geometry classes and the more advanced courses in the best schools continued to teach a curriculum very much like the new math curricula of the 1960s.

Concurrent with the back-to-basics movement came a movement for minimum competence, and these two movements together had the effect of encouraging teachers to teach algebraic skills without understanding and to lessen attention to proof in geometry. There was also a positive effect: Books were cleansed of the excesses of the new math. For instance, the ubiquitous first chapter on sets that had little relation to the rest of the book was taken out, as were overzealous formalisms and explanations that were at too high a level for student understanding.

Most parents of today's students took the courses in the 1970s that their children are being taught now. So the experience of the parents of current students is likely to have been at the time when new math was being branded a failure and back-to-basics curricula were being touted.

It is difficult to find any value in the back-to-basics backlash other than the cleansing of the excesses of new math. Within a few years following the backlash, scores on the SATs were the lowest they have ever been. That situation prompted quite a number of reports in the late 1970s and early 1980s encouraging improvement in mathematics education. Some of these reports promoted problem-solving rather than skill development as the key goal of school mathematics. Others promoted a rethinking of the high school curriculum with continued attention to algebra, geometry and functions, but stronger attention to applications, to probability and statistics, and to the widespread use of calculators and computers and the mathematics related to them. Mathematicians and mathematics educators worked together on these reports. They were, for the most part, not the same people who had led the new math movement of 25 years earlier. The mathematicians included applied mathematicians, computer scientists, and statisticians. The mathematics educators included big city and state supervisors. The most well-known of these reports was *A Nation at Risk*, which appeared in 1983.

THE CURRENT ERA

The situation since 1983 has been strikingly parallel to that of the new math era. The first of the reform projects was UCSMP. Six years later the major catalyst for more reform appeared in the form of the NCTM *Curriculum and Evaluation Standards for School Mathematics*. Within a couple of years, the government—specifically, the National Science Foundation—poured massive amounts of money into curriculum reform. These events followed almost exactly the schedule of the development of the new math 32 years earlier, as Table 1 indicates.

Again there was euphoria. State after state adopted its own version of the *Standards*. NSF felt so good about its projects that it assumed they would be successful and planned for dissemination well before any data were collected. And there are statistics to back up these good feelings about the current era. During

Table I
Parallel Developments in New Math Era and Current Era

	New Math Era 1951-1973	*Current Era* 1983-
First project: year n	UICSM	UCSMP
Catalyst for more projects: year n+6	Sputnik	NCTM *Standards*
Government help years n+7 on	NDEA	NSF curricula
Sign of euphoria year n+11	Cambridge Conference	States follow NCTM
False signs of failure years n+14, n+21	SAT decline, NAEP	TIMSS
True sign of failure	Poorer students lost	not known

the 1990s, more students have taken more mathematics in high school than ever before. Until a decline of a single point this year, in every year of the 1990s SAT scores have stayed the same or increased from the previous year. ACT scores have also either increased or stayed the same for each year in the decade. Mean scores on the long-term trend data of the National Assessment of Educational progress have increased. It has been a decade of phenomenal growth.

However, again there is a false sign of failure. This time it is the misinterpretation of the results from the Third Intenational Mathematics and Science Study (TIMSS). The TIMSS researchers did not compare performance of our students now with the performance of our students on FIMS (First International Mathematics Study, 1964) or SIMS (Second International Mathematics Study, 1981). If they had, the headlines would have been different, because U.S. students seem to have performed quite a bit better comparatively on TIMSS than on the previous studies.

> “
> ***The disparities are tragic...even though, ostensibly, we do not teach different mathematics in these different places.***

Specifically, the U.S. is being compared to Singapore, which scored even higher than Japan at the 4th and 8th grade levels. (Singapore did not participate at the 12th grade level.) But I will argue that the U.S. performs strikingly well, even compared to Singapore. My argument has to do with economics, sociology and geography.

First, the economics and the sociology. Throughout the world, both FIMS and SIMS showed that performance within a country was higher in those places within the country that were more affluent. The one exception to this was Japan, where performance was quite uniform throughout the country.

It is well-known that performance within the U.S. fits the international pattern. That is, throughout our country the best performing students in general are found in our affluent suburbs and the lowest-performing students are found in our poorest rural and urban areas. In our affluent suburbs, the students do score as well as the students from Singapore. Our evidence for this comes from the performance of students in the First in the World Consortium outside Chicago on TIMSS. I am reasonably certain that performance would be matched in similarly affluent places elsewhere in the country where the schools can select their own curriculum and are not subject to state constraints. Just this week, Gerald Bracey, a writer for *Phi Delta Kappan* on the interpretation of educational research, has reported that such a study has been done of the data from TIMSS and that it shows our suburban areas would be second in the world. If true, it would indicate that these students score as high despite many of the students not having a curriculum that is as advanced as that of Singapore. It would thus show that our suburban students learn better what they are taught than students from Singapore.

Now for the economics and the geographic part of the argument. Singapore is an independent country, but viewed from a larger geographic perspective it is the most affluent area of southeast Asia. Its per capita gross national product is five times higher than that of Malaysia which surrounds it and is only surpassed by the very small country of Brunei. Singapore’s per capita GNP is higher than that of Spain or Hong Kong or New Zealand. As with our suburbs, in recent generations people migrated to Singapore from neighboring areas, mostly China, because they wanted a better life. Today over three-fourths of the population of Singapore is Chinese even though Singapore does not lie close to China. The population of Singapore is special for the same reason that the population of our suburbs is special. And for these reasons the performance is similar.

There is no question we can do better than we have been doing. The disparities are tragic between performance in some states and others, and between performance in our suburbs and our cities, even though, ostensibly, we do not teach different mathematics in these different places. The performance in our more affluent areas demonstrates that we can improve what we are doing without major changes in curricula, but it also suggests that we might have to change economic opportunity in order to do so. We still have huge numbers of mathematics teachers who do not know enough about the subject to teach it well. These teachers have trouble handling a curriculum like UCSMP's which wants students to have more than one way of doing a problem and asks students to apply mathematics and make connections.

But is there a sign that we are doing worse than we have done in the recent past? I don't know of a single national study in which such a signal is found. Furthermore, states such as North Carolina and Texas and Michigan, whose National Assessment results have increased the most of any states in the country, are those who claim to have implemented the current kinds of reforms. Nevertheless, there are those who claim that the present reforms are a failure.

BELIEFS OF THE ANTI-REFORMERS

While a great number of mathematicians support the reforms in K-12 mathematics education, another group opposes these reforms. The anti-reform mathematicians are from the same types of outstanding universities as the mathematicians of the Cambridge Conference. For the most part, they are research mathematicians. Some are quite eminent. We cannot expect their knowledge of mathematics education and of students in schools to be any greater than that of the mathematicians in the Cambridge Conference. But, unlike the Cambridge Conference mathematicians, who took their role to be provokers and were careful to say that their ideas needed to be tested, these mathematicians desire to directly affect mathematics education.

In one state of the union they have taken over, and from this state we can obtain a picture of the solution these mathematicians offer. Their solution is found in the *Mathematics Framework for California Public Schools*.

The catalyst for the *Mathematics Framework for California Public Schools* was California's poor performance on the 1996 National Assessment of Educational Progress. California scored 3rd lowest of the 44 states that participated in this assessment test. Its mean scale score of 138 was 10 points, or approximately one full grade level, behind the national norm of 148. But this disguises the differences among the performance of various subgroups. White students in California were only 3 points below the national mean for white students. Asian students scored only 2 points below the national mean for Asian students. Black students scored 1 point above the national mean for Black Students. But Hispanic students, constituting 39% of the student population, scored 27 points behind the total national norm and 6 points behind the national mean for Hispanic students, and they caused the state's overall mean to be so low compared to the nation. [*Science and Engineering Indicators*, p. A-12]

With such diversity, it would seem reasonable to leave decisions to local school districts about what mathematics should be taught. But there is a history in California of strong control from the state's Department of Education in Sacramento. For grades K-8 the state approves books, and the approved books must follow the state framework. *Mathematics Framework for California Public Schools*, therefore, is not just a theoretical document; it has teeth.

Twenty-four individuals are listed as having contributed to the present *Mathematics Framework*. Not one of these individuals is a university mathematics educator, and all the sample problems were developed by university mathematics professors.

The tone of the document reflects the excesses rather than the lessons of the new math. Here is the introduction to one of those problems [p. 154]: "Starting with grade eight, students should be ready for the basic message that logical reasoning is the underpinning of all mathematics...Students should begin to learn to ***prove*** every statement they make. Every textbook or mathematics lesson should try to convey this message and to convey it well. Consider the problem of solving this equation:

$x - \frac{1}{4}(3x - 1)= 2x - 5$

Multiply both sides by 4 to get:

$4x - (3x - 1) = 8x - 20$

Then simplify the left side to get:

$x + 1 = 8x - 20$

Transposing x from left to right yields:

$1 = 7x - 20$

One more transposition gives the result $x = 3$.

So far this seems to be an entirely mechanical procedure. No proof is involved."

No proof is involved? It looks very much like a proof to me, except that I would emphasize doing the same things to both sides of the equation and avoid words like "transposing" that suggest to students that math-

Recommended Proof of $x - \frac{1}{4}(3x - 1) = 2x - 5$ by the writers of the Mathematics Framework

1.	$x - \frac{1}{4}(3x - 1) = 2x - 5$	1.	Hypothesis
2.	$4(x - \frac{1}{4}(3x - 1)) = 4(2x - 5)$	2.	$a = b$ implies $ca = cb$ for all numbers a, b, c.
3.	$4x - 4(\frac{1}{4}(3x - 1)) = 4(2x) - 20$	3.	Distributive law
4.	$4x - (4 \bullet \frac{1}{4})(3x - 1) = (4 \bullet 2)x - 20$	4.	Associative law for multiplication
5.	$4x - (3x - 1) = 8x - 20$	5.	$1 \bullet a = a$ for all numbers a
6.	$4x + (-3x + 1) = 8x - 20$	6.	$-(a - b) = (-a + b)$ for all numbers a, b.
7.	$(4x + (-3x)) + 1 = 8x - 20$	7.	Associative law for addition
8.	$x + 1 = 8x - 20$	8.	$4x + (-3x) = (4 + (-3))\, x$, by the distributive law
9.	$-x + (x + 10) = (-x + 8x) - 20$	9.	Equals added to equals are equal.
10.	$(-x + x) + 1 = (-x + 8x) - 20$	10.	Associative law for addition: $0 + 1 = 1$.
11.	$1 = 7x - 20$	11.	$-x + 8x - (-1 + 8)x$, by the distributive law
12.	$1 + 20 = (7x - 20)+ 20$	12.	Equals added to equals are equal.
13.	$21 = 7x + [(-20) + 20]$	13.	Associative law for addition
14.	$21 = 7x$	14.	$-a + a = 0$ for all a; $b + 0 = b$ for all b.
15.	$3 = x$	15.	Multiply (14) by $\frac{1}{7}$ and apply the associative law to $\frac{1}{7}$(7x).
16.	$x = 3$	16.	$a = b$ implies that b = a Q.E.D.

Mathematics Framework for California Public Schools, p. 155

ematics is a bag of tricks. If you showed that 3 is indeed a solution to the first equation, then I would argue that this is a proof that $x - \frac{1}{4}(3x - 1) = 2x - 5 \iff x = 3$. But this does not constitute a proof for the *Mathematics Framework* writers. Their proof can be found at the top of the next page. It is, in my opinion, cruel and unusual punishment to inflict this kind of pedantry onto young children.

Under the criterion for proof used by the Mathematics Framework writers, virtually every argument labeled a "proof" in any college textbook or any article on mathematics would be disqualified. So we now have a new criterion for a written proof. It must be rigorous. But even the presented proof is not rigorous. A couple of reasons are missing in step 15. And where are the logical principles such as modus ponens and the transitivity of implication?

The authors of this example have confused rigor and proof. They have confused logic and proof. And in the process they have repeated one of the major excesses of the new math era: the overemphasis on rigor.

There is a significant marginal comment on this page. "Without the realization that a mathematical proof is lurking behind the well-known formalism of solving linear equations, a teacher would most likely emphasize the wrong points in the presentation of beginning algebra." I agree with the point that students should learn that solving an equation proves a statement. But this is not the time to learn that. The authors have made a natural but fundamental error about teaching young students. Every teacher learns through experience that students learn in different ways and that a multitude of explanations are needed, ranging from the formal to the intuitive, from the symbolic to the pictorial.

Because the countries that scored highest on TIMSS tend not to use calculators in early grades, the authors of this framework conclude that calculators cause our students to perform poorly. Having asserted that correlation does not imply causation, they reason as if it does. The assessment program that goes along with this framework does not allow the use of calculators from kindergarten to grade 11. The authors ignore the fact that Singapore, Japan, and China, in their newer elementary curricula, are introducing calculators because they have come to realize the necessity of their students being technologically facile with mathematics. If you are interested in reading about these international developments, examine the proceedings from UCSMP's Fourth International Conference on Mathematics Education held in the summer of 1998 and now available in *Developments in School Mathematics Education Around the World, Volume 4* from NCTM.

There are a number of applications presented in the *California Framework*, particularly in the sample problems. Statistics has a strong presence. But modeling, as essential to applied mathematics as proof is to pure mathematics, is completely absent. The student will leave high school not realizing that mathematics is applicable outside of money matters, statistics, and the physical sciences.

With the exception of statistics, the *Mathematics Framework* ignores virtually all of the developments in the mathematical sciences in the past 50 years. The authors have created a curriculum that asserts what was good for students 30-40 years ago is still what's good for students today. And they have not taken into account that such a curriculum destroyed students at the bottom end.

CONCERNS AT THE COLLEGE LEVEL

Anti-reform mathematicians appear to be motivated by three major concerns. A first concern is that, at the top end, we are not creating enough students with high mathematical competence. This includes the concern that we are not creating enough well-trained mathematics teachers. A second concern is that too many students enter college needing to take remedial mathematics courses because they lack sufficient paper-and-pencil manipulative algebraic skills. The third concern is the decreasing emphasis on proof in secondary school mathematics courses.

The third concern, that proof is disappearing from high school mathematics, is one that we in UCSMP feel is a valid concern. We have tried to incorporate proof into four of our courses, with strong attention in both Geometry and PDM.

Most of my data about the first concern, students at the top end, come from the National Science Board report entitled *Science & Engineering Indicators* for 1998. This means that the data go no further than 1996. Let us begin with AP calculus.

Advanced Placement Students

In 1994 7% of all high school graduates took an AP calculus course, compared with 4% in 1990, 3% in 1987, and 1.5% in 1982. Almost half of these students are female. [*Science & Engineering Indicators*, 1998, p. A-16] This enormous increase is due in part to increasing numbers of students taking algebra in eighth grade, a trend in which UCSMP has had a hand. Even the birth of AP statistics has not lowered the numbers of students taking AP calculus.

Percent of Students Intending to Major in Science or Engineering

I could not find data for all students, so these data are limited to white students only. In 1996 32.2% of white freshmen intended to major in science and engineering. This is the highest percent in the last twenty years. Of these, 11.6% planned to major in the natural sciences. That is lower than the 12.0% of 1995 and the 11.9% of 1994, but higher than every other year since 1976.

The percent of white freshmen planning to major in mathematics or computer science peaked in 1982 at 5.9% but by 1985 had gone down to 2.5% and by 1992 was 2.2%. Since 1992 there has been a reasonably steady rise in this percent. In 1996 2.7% of freshmen planned to major in mathematics or computer science, the highest percent since 1985. [p. A-57]

Table 2
Bachelor's Degrees in Mathematics and Computer Science Awarded in the US Between 1975 & 1995

	Mathematics	*Computer Science*
1995:	13,851	24,769
1987:	16,515 (relative max.)	39,927
1986:	16,388	42,195 (max.)
1981:	11,901 (relative min.)	15,233
1975:	18,346	5,039

From *Science & Engineering Indicators 1998*, p. A-64.

The net result is that more students are now coming into college with high-end mathematics and with a broad desire to major in science or engineering, and a specific desire to major in mathematics or computer science, than in the 1980s before the current reforms. Whether this is due to economics or to curriculum, I do not know.

Now let us ask what happens to these students.

Number of Bachelor's Degrees

The number of bachelor's degrees in mathematics has seesawed. It was 18,346 in 1975, a year in which most students would have had their high school education in new math-oriented curricula. From 1979-1984, years in which students would have had their high school education affected by back-to-basics, it went down to around 12,500. Since 1985, its peak was in 1987 and there has been a steady decline to the 1995 level of 13,851, which is about 20% below the 1987 level. (See Table 2.) This is a serious problem, because we need more mathematics majors, but it will have been solved by this year if the degree-intending students of 1996 get degrees in mathematics proportional to their numbers in prior years.

Even more surprising is a much more serious decline in the number of bachelor's degrees in computer science than the decline in mathematics. There were over 42,000 computer science degrees in 1986 but under 25,000 in 1995. (See Table 2.)

Foreign Citizens

There has been a reasonably steady increase in the percent of bachelor's degrees in mathematics or computer science given to foreign citizens, from 5% in 1985 to 7% in 1995. [p. A-67] It is very difficult to see how this is related to changes in the U.S. curriculum. It seems far more related to the easing of world tensions early in this decade, to the increase in the study of English worldwide as a second language, and to the increased desire for a college education by people in countries where college attendance is not as accessible as in the U.S.

Graduate Enrollment

Graduate enrollment in mathematics and computer science is double what it was in 1975 and has fluctuated in a narrow range since 1987. The percent of foreign nationals has also fluctuated, between 26% and 33% since 1983, peaking in 1991. It is currently at 32%. [pp. A-70, A-72] Graduate enrollment has to lag quite a bit behind school curriculum changes, so these data are not influenced by the current reform movements.

Table 3
Freshmen Reporting Need for Remedial Work in Mathematics in 1995

By major and gender

	All	Male	Female
Physical science:	13.0%	11.3%	15.5%
Engineering:	14.3%	13.9%	15.5%
Social science:	27.5%	20.2%	32.2%
non-science or eng:	25.0%	20.8%	28.0%

By major and ethnicity

	White	Black	Hispanic
Physical science:	11.1%	39.9%	20.7%
Engineering:	10.3%	31.1%	34.1%
Social science:	24.2%	47.5%	41.2%
non-science or eng:	22.0%	47.1%	37.0%

Doctoral Degrees

The number of doctorates in mathematics in 1995 was 1,190, the highest it has been since 1975 and 72% higher than its low value in 1985. The number of doctorates in computer science in 1995 was the highest it has ever been. While the total number of doctorates in mathematics and computer science granted to temporary residents of the U.S. has more than doubled since 1977, the number of doctorates granted to U.S. citizens has also increased significantly. [p. A-82]

These data do not suggest a crisis, and certainly not anything attributable to recent curricula.

Now let us consider the second concern—that students are more poorly trained than they used to be. I do not have long-term data on this, but 13% of all physical science and 14% of all engineering freshman majors in 1995 reported a need for remedial work in mathematics. This figure varies markedly with ethnicity: 11% of whites but over 30% of Blacks. In other words, 1 out of 9 white students and 1 out of 3 Black students majoring in the physical sciences or engineering reported a need for remedial work in mathematics. (See Table 3.)

Significantly more freshman majoring in social science need remedial work in mathematics: 20% of males and 32% of females; 24% of whites and 48% of Blacks. These high percentages seem to reflect the ancient view that if you are going to major in the social sciences, you don't need to take as much mathematics in high school. These students, generally not as proficient, will not be helped by a more theoretical mathematics curriculum.

I assume that these percents are higher today than they used to be. And so we must ask: if scores of high school students are going up, why are the percents of students needing remediation not going down?

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> ***They may not be as good as most mathematics students were in the past, but they are as interested, and we need them because we have a chronic shortage of mathematics teachers.***

The reasons are many. First, despite revolutions in applied mathematics and in the ways in which computers change how mathematics can be done, the mathematics departments of many, if not most, major universities have not changed their basic required curriculum in a generation. And, because they haven't changed their curricula, they haven't changed their tests to represent what students are taught in high school. So the students do not score as well as they used to score (though, frankly, I have yet to see a published study on college placement tests over time). Mathematics departments need to wake up and recognize statistics, computer science, and applied mathematics as topics that are as important for mathematics majors as algebra and analysis, and test incoming students on their knowledge of basic ideas from these areas. Placement exams need to recognize that computers are here to stay, and allow students to use calculators on the tests because they will have such calculators with them their entire lives.

A second reason for more remediation is that college mathematics requirements have increased. Fields that used to require very little mathematics—psychology, business, the biological sciences, and the social sciences—now require statistics or calculus and sometimes linear algebra and finite mathematics. Many institutions now require some mathematics of all their students. In this regard, high school counselors are often behind the times. And, consequently, some students come to college without having taken the mathematics they should have taken to major in these fields. And, when they have taken the mathematics, they thought that it was just to fulfill an entrance requirement and did not realize that they would have to demonstrate competence.

Third, high schools are doing a better job of interesting their students in mathematics, and so some students who are not the very best students still like the subject, and they want to major in it even though they will not be research mathematicians. They may not be as good as most mathematics students were in the past, but they are as interested, and we need them because we have a chronic shortage of mathematics teachers.

Mathematics departments in many institutions operate as if computers and calculators do not exist, have requirements that suggest that applications of mathematics are for the not-so-serious student, still think that writing logically-correct mathematics legibly is sufficient to be called good teaching. No wonder that they are losing students to statistics, to economics, to operations research, to business and to many other disciplines which yearn for students who like and are proficient at mathematics. Who wants to major in an area that ignores even major changes within it?

Some people have argued that baseball players are not as good today as they were fifty years ago. Fifty years ago if you wished to be a professional athlete, you had little choice but to go into baseball. So baseball got the best athletes. There may be an analogy with mathematics. Fifty years ago if you liked pure mathematics, your choice was limited to mathematics. But today, you can go into many disciplines where your talents will be utilized.

A fourth reason for the poor performance of incoming college freshmen on placement tests is that placement tests are often given under conditions that do not allow students to show off what they know. A test given to students who have just come to campus a few days before, who are concerned about their new roommates, about the medical exam they just had, about their new ID card, not to mention being away from home, and who may have stayed up quite late talking to others in their dorms, is not being taken under optimal conditions. Also, students have had different courses in high school and need to be informed in advance exactly what topics are going to be on the test, and the kinds of language and notation that are going to be used.

REASONS FOR DISAGREEMENT

Why do people come to different conclusions about what is happening? Why does a U.S. Department of Education panel's choice of the best mathematics materials in the country for grades K-8 have nothing in common with the books selected earlier this year in California, except for UCSMP Transition Mathematics and Algebra?

We who are in the field are privy to much information about what is going on. We may be aware of the politics on both sides. But what are parents and the public to think? When there are conflicting views about an issue, and there seems to be no overwhelming authority, the tendency is to believe the loudest or the boldest. The press does not help; they revel in publicizing conflicts and tend to select individuals with extreme positions to make the point that there is a conflict.

We who are in mathematics must fight this tendency, regardless of how we feel about the issues. Truth in our field is based on careful reasoning. If we are in pure mathematics, we reason from assumptions using logical deduction. If we are in applied mathematics, we analyze data using statistical principles. In neither case should we allow untested opinion to sway us. In those cases where we do not have enough evidence to make a conclusion, we should be willing to say that a problem is unsolved. If we come to different conclusions, we ought to try to apply the tools of mathematics to determine why.

I don't think the critics of current reforms are operating with the same assumptions that we have, and I would like to finish by asserting some of the assumptions under which we operate at UCSMP. We in schools must educate everyone, and we cannot assume our students are motivated by the same things that motivate university-level mathematicians. As the NCTM Professional Teaching Standards emphasize, teaching is a complicated process, not subject to simple prescriptions. In some cases logical approaches work, but for many topics a good application or a game or an activity works better, and representations can be particularly powerful. Capable mathematics teachers who teach students every day contributed to the NCTM Standards and the newer curricula. They are not ignorant of mathematics. We want our students to have the same appreciation for its beauty, its logical structure, and its applications that we have. We try to instill in our students an appreciation also for careful reasoning, for not assuming a conclusion without weighing all, of the evidence. Statistics and mathematical modeling help our students to weigh data, to recognize the importance of comparable samples when comparing groups, to realize that there may be more than one answer to a real problem. We teach the students of today for what they need tomorrow, not for what they needed yesterday, and we realize that to avoid the use of technology is to doom our students to ignorance of much of the world of mathematics. We recognize that mathematics is important in consumer affairs, in matters of public policy, and in business as well as in its traditional venues of science and engineering and a subject to study for its own sake. It is because mathematics is more important than ever that we must work to see that all students are not only taught a significant amount of mathematics, but that they learn it.

"Try not to become a man of success but rather to become a man of value."

--Albert Einstein

Book Review: *The Teaching Gap*
by James W. Stigler and James Hiebert
Review Part I: Mathematics Teaching as a Cultural Activity

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The Teaching Gap. James W. Stigler and James Hiebert. The Free Press (Div. Simon & Schuster Inc.): New York, 1999. ISBN 0-684-85274-8

As any mathematician who has worked widely in applied statistics knows firsthand, it is true in the social sciences no less than the natural sciences that new discoveries often begin with the availability of new kinds of data. And such data, in the social sciences, in turn often owe their existence to novel ways of harnessing, to the purposes of science, technology for recording human behavior. Virtually whole new fields of study may be born in this way: a piquant example is the field of child language acquisition, which burgeoned with the availability of portable audiotape recorders in the 1970s.

The Teaching Gap is based on novel, indeed unique, data of high quality and unprecedented scope: a random sample, statistically controlled to minimize bias, of eighth-grade mathematics classroom lessons in the U.S., recorded on videotape, and corresponding information from Japan and from Germany—three national video samples, representative of teaching in each country. The video study "is the first to collect videotaped records of classroom instruction—in any subject—from national probability samples."[1]

This video study is actually one component of the Third International Mathematics and Science Study (TIMSS), itself as a whole a much more advanced study methodologically than its like-named predecessors. "Fortunately, the TIMSS sampling plan was highly sophisticated...the video sample [was] a random subsample of the full TIMSS sample. Not only were specific teachers selected, but specific class periods as well. No substitutions were allowed..."[2] It is evident that substantial efforts were made in the video study (and in TIMSS as a whole) to minimize both sampling and nonsampling errors.

The last quote is from an excellent overview article on the video study, written by the authors of *The Teaching Gap*, and available on the Web.[3] The authors maintain a Web site[4] with links to articles, including this one, that are related to *The Teaching Gap*, the video study, and TIMSS. One of the links (still under construction as of this writing) from this site will allow the viewing of some of the actual videos.

The decision by the National Center for Education Statistics to collect national videotape samples reflects the early influence[5] of one of the authors of *The Teaching Gap*, James W. Stigler, who was a co-author of an earlier highly regarded study[6] comparing Japanese and Chinese education to our own. Stigler realized that the problem of a lack of common understanding of basic terms to describe teaching, all the more serious in an international context, would require data more raw, less filtered, than questionnaire responses could offer.[7]

Based on these video data, the authors set out to address fundamental questions about mathematics teaching as it is actually and typically done, in the three countries: What methods do teachers use to teach? Does mathematics teaching differ in any significant ways from one country to another, or are teachers in all three cultures teaching mathematics more or less the same way? And, in the U.S., are high-profile reform recommendations actually being put into practice? Also, because the video data would show actual classroom teaching as it is, unmediated by measurement instruments such as questionnaires, such data would have the power to surprise and to reveal the unexpected.

And surprise they did: "To put it simply, we were amazed at how much teaching varied across cultures and how little it varied within cultures."[8] This core finding of the book, stated here in formal language reminiscent of a statistical analysis of variance, ought to cause a double-take: what is being said here seems important and fundamental, even stunning. With the authors' meticulous support, both qualitative and quantitative, this finding gives an empirical basis in the case of mathematics teaching to the claim which forms the title of the pivotal and most profound chapter in the book (Chapter 6): "Teaching is a Cultural Activity."

And this finding, with the evidence for it, is the central reason this review is being offered in a journal on humanistic mathematics, one essential concern of which we take to be how the expression of mathematical activity is shaped by its cultural backdrop.

Of course this empirical basis extends only to the teaching of mathematics, since all the data in the video study are from mathematics classes. To those of us whose main concern is the teaching of mathematics anyway, this is not of great consequence. But it is an inferential lapse, that mars the book's otherwise careful methodological presentation, to claim (as the authors implicitly do throughout the analysis of the video study) that the "points we make go well beyond mathematics" and thereby to extrapolate a conclusion well beyond the scope of the data. Seeing the authors' findings, one is certainly inclined to hypothesize that teaching in general *is* similarly culturally conditioned, but the authors present no proof of that general proposition.

A related extrapolative claim that the authors make is indirectly better supported, however, namely that the points they make extend "certainly well beyond eighth grade." Indeed, there are enough commonalities even with our own experiences at the college level, as we suggest below and in Part II, to make this claim persuasive.

It is widely known that American educational achievement (not only) in mathematics does not stack up well in international comparisons; e.g., TIMSS showed this, and did so even more authoritatively than did its two antecedent international studies. Nor is it any secret that wave after wave of efforts to reform American education (not only) in mathematics has failed to result in improved student performance.

For this dismal record the authors have a simple yet profound core explanation, one that reverberates like a theorem understood for the first time, feeling like something we have known liminally all along: in our efforts to reform American education, "We have been acting as if teaching is a noncultural activity." But teaching is a culture-bound activity, and this "explains why teaching has been so resistant to change," and our not taking that fundamentally into account is why we have failed. (Again, as noted above, the authors overreach their empirical base, which is only in mathematics teaching, but the reader is inclined to go along.)

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> ***Teaching is a culture-bound activity, and this "explains why teaching has been so resistant to change," and our not taking that fundamentally into account is why we have failed.***

Taking this fundamentally into account is also why the Japanese, in particular, have succeeded: "In Japan, by contrast, teaching practices appear to have changed markedly over the past fifty years." And Japanese students, correspondingly, now perform among the top internationally. The authors look, for a model, to the Japanese system for the improvement of teaching not only for these reasons but also, and crucially, because "Japan's system of improvement ... is built on the idea that teaching is a complex, cultural activity."

The Teaching Gap's analysis and interpretation of the video data work up to the chapter mentioned above, entitled "Teaching is a Cultural Activity," and substantiate this claim. This analysis is concerned with the details of what actually goes on in classrooms and characterizes teaching in the three countries, showing both qualitatively and quantitatively the stark inferiorities of U.S. mathematics teach*ing* (not teachers) to the other countries', especially Japan's. This material can raise our awareness of features of our teaching that we might well have been taking for granted, and offer fertile images of alternatives.

The authors follow this analysis by proposing a mechanism for slow, organic change of our teaching methods, based on the system in Japan whereby teachers, acting as researchers, seek to improve classroom teaching, lesson by lesson. This proposal speaks to our concerns for the preparation of our future students, as well as our sense of responsibility for the improvement of mathematics education in the public schools.

What then does teaching look like in the U.S., Japan, and Germany? To show this qualitatively and as an image, the authors synthesize the video data into a single typical pattern for each country's mathematics lesson. We focus on the U.S. and Japanese patterns, since these exhibit the most stark contrast.

Both the Japanese and U.S. lessons typically began with a review of previous work. In the U.S. this was followed by "presenting a few sample problems and demonstrating how to solve them." Then the students practiced solving problems like those presented. Finally, there was checking and correcting some of the students' practice work (and assigning homework). In Japan, the initial review was followed by the presentation of a new problem for the day's lesson. Students then worked on trying to solve the problem. There followed a discussion of various methods of solution that the students had come up with or that the teacher showed them. The lesson ended with the teacher emphasizing the main points.

Thus, while within-culture variation (such as different ways to demonstrate a procedure) looked so large when the authors watched only U.S. lessons, when they "watched a Japanese lesson...we noticed that the teacher presents a problem to the students without first demonstrating how to solve the problem. We realized that U.S. teachers almost never do this, and now we saw that a feature we hardly noticed before is perhaps one of the most important features of U.S. lessons—that the teacher almost always demonstrates a procedure for solving problems before assigning them to students." Thus, while both systems have the presentation of a new problem, this activity in Japan is preparation for students to develop solution procedures, while in the U.S. it allows a procedure to be demonstrated and is followed by students practicing the procedure.

This is, I think, the most critical single observation in the book. I find it thought-provoking indeed to reflect on how this contrast may also fit our U.S. mathematics teaching at the college level.

The authors accordingly find the teaching of mathematics in the U.S. to be very constricted, "focused for the most part on a very narrow band of procedural skills." Regardless of whether students are working individually or in groups, or whether they are using computers, American mathematics students "spend most of their time *acquiring isolated skills through repeated practice*" [italics ours]. "Japanese teaching... [shows] what it can look like to teach mathematics in a deeper way, teaching for conceptual understanding. Students in Japanese classrooms spend as much time solving challenging problems and discussing mathematical concepts as they do practicing skills."

To support these qualitative generalizations, the authors characterize lessons in each country with statistics on various salient features—"in research parlance, 'indicators'—that might influence students' learning." The U.S. lessons fall correspondingly short on these quantitative summaries. The percent of U.S. vs. Japanese lessons respectively exhibiting "concepts developed rather than [merely] stated" was 22% vs. 83%; "medium [or] high quality of mathematical content" (as opposed to low quality of content) was 11% or 0% vs. 51% or 39%. The "average percentage of seatwork time spent in ... apply[ing] [or] invent[ing]/think[ing]" (as opposed to merely "practic[ing]" in familiar contexts) was 3.5% or 0.7% vs. 15% or 44%. The list goes on.

Among its uses, this list can serve as a brief but salutary cold shower for any of us at the college level who may be entertaining misattributions as to what our students' preparations might be like. "[T]here were *no* mathematics proofs in U.S. lessons. In contrast, there were proofs in 53 percent of Japanese lessons..." [italics theirs].

Probably more importantly, we can evaluate our own teaching by the criteria in this list and reflect on how the results of this evaluation might follow from the critical observation contrasting U.S. and Japanese teaching cited above.

Part II of this review of *The Teaching Gap* resumes with a consideration of teaching as a tightly interconnected *system* of features and the underlying cultural *beliefs* that vivify such a system in the U.S. and in Japan. There are some surprises here that have relevance to college classrooms.

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[8] This quote, along with all others not footnoted, is from *The Teaching Gap.*

Funny Problems

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A selection of original or collected recreational mathematical problems.

1) Prove that 2 = 1
Solution:
2 pints = 1 quart.

2) A man weighs the following weights on the following dates. How is this possible?

6/1/70	150 lbs.
6/3/70	0 lbs.
6/5/70	25 lbs.
6/7/70	0 lbs.
6/9/70	145 lbs.

Solution:
The man is an astronaut who went to the moon and back.
Outerspace weightlessness: 0 lbs.
$\frac{1}{6}$ of Earth's gravity, or gravity of the moon: 25 lbs.

3) If you have a couple of threes and divide them in half, why do you end up with 4 pieces?
Solution:
33 cut in half horizontally will make four pieces.

4) How 70 > 3 = LOVE?
Solution:
Move the characters of 70 > 3 around.

5) 10 - 1= 0
Solution:
If you have a stick (1) and an egg (0) and you give away the stick (1) you still have the egg (0) left.

6) All monkeys eat bananas.
I eat bananas.
Therefore, I am a monkey!

7) Twelve minus one is equal to two.
Solution:
12 - 1 = 2 (take digit 1 from 12).

8) 7 + 7 = 0.
Solution:
Take the sticks from the 7's and rearrange them to form a rectangular zero 0.

9) 3 x 2578 = hell
Solution:
Read your calculator upside down: 7734 (the product of the numbers) becomes hell (approximately).

10) An earthworm is cut down the middle. How many halves are there?
Solution:
One, because the other half can still be one whole earthworm.

11) From two false hypotheses get a true statement.
Examples:

a)	Grass is edible.	(False)
	Edible things are green.	(False)
	Therefore, grass is green.	(True)
b)	All dogs are poodles.	(False)
	Spot is a dog.	(False)
	Thus, Spot is a poodle.	(True)

12) How can you add 3 with 3 and get 8?
Solution:
Turn one of the threes around and put them together to make an 8 (approximately).

13) If 10 trees fall down, and no one is around to hear them falling, how many of the trees fall?
Solution: Ten.

14) When algebraically 1=0?
Solution:
In a null ring, which is a set with only one element and one binary operation. If we take for "+" and for the same operation, we get a commutative unitary ring.
In this case, the unitary element for "*" (which is normally denoted by "1") and the null element, (which is normally denoted by "0") coincide.

15) When is it possible to have: 1 + 1 = 10?
Solution: In base 2.

16) Another logic:
How can we have ten divided by two equal to zero?
Solution:
Ten cookies divided by two kids are eaten and nothing remains!

17) You are lost and walking down a road. You want to get to town and know the road leads to town but don't know which direction. You meet two twin boys. You know one boy always tells the truth and one always lies. The boys know the direction to town. You cannot tell the boys apart and can only ask one question to one boy to find the direction to town. What question should you ask?
Solution:
Ask either boy what the other boy would say is the direction to town. This would be a lie because if you were asking the dishonest boy he would tell you a lie. If you were asking the honest boy he would tell you the truth about what the dishonest boy would say (which would be a lie) so he would give you the wrong direction. Town would then be in the opposite direction.

18) Why are manhole covers round? You know, the manholes on the streets, is there a reason why they made them round or could they be square or triangular?
Solution:
Manhole covers are round because a circle cannot fall inside of itself. If they were square, triangular or some other shape they could be dropped into the hole, which would be dangerous to traffic.

19) You have eleven lines. How can you move five lines and still have nine?
Solution:
| | | | | | | | | | | —> | \ | | | \ | E

20) You have a cannon and two identical cannon balls. You take the cannon to a large open location that is perfectly flat and you adjust the cannon barrel so that it is perfectly level. You load one of the cannon balls into the cannon and you hold the other cannon ball at the same height as the barrel. You fire the cannon and drop the other cannon ball at the same time. Which cannon ball will hit the ground first?
Solution:
Both cannon balls should hit the ground at the same time, since gravity acts equally on two objects having the same mass. The cannon barrel was leveled and the cannon ball would begin to fall as it moved forward out of the barrel at the same rate as the cannon ball that was dropped by hand. They would hit at the same time but the cannon ball fired from the cannon would hit the ground far away.

Students and Their Learning from Reading

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INTRODUCTION

My aim in this article is to summarize work I have done over the last three years, focusing on the issue of helping students learn from whatever mathematics text they read. Although these types of texts generally contain 3 modes of communication, namely technical English, the language of mathematics itself and diagrams, I will focus this article only on the technical English of such texts. The idea, then, that students can develop techniques and strategies for learning from what they read is generally known as "reading to learn."

AN OVERVIEW

Now, mathematics courses cover a variety of subjects from statistics and O.R. to pure mathematics to applied mathematics. I mention this only to point to the fact that the textbooks students use to read, and hopefully learn from, are written in such different styles and contain such depth of detail that they have great problems developing an understanding of what they read. Statistics texts tend to be written in a more prosaic and descriptive style than that of applied or pure mathematics texts which usually tend to be very tightly structured in terms of language, containing a high concentration of technical words.

Given this, and the fact that students spend more time trying to learn from written material than having access to a teacher who can support them in their learning, it might be beneficial for them to be able to learn how to go about reading meanings into the texts they read, and as a result learning from these. Consequently, the realm of reading-to-learn part of my work has focused on developing techniques which allow students to develop an ability to read to learn from text written in plain or technical style of language by adopting an interpretive approach to their reading. Supporting students' learning from reading is done via the use of text manipulation and gist elicitation techniques aimed at allowing students to develop their own personally significant meaning and understanding of the text.

INTERPRETATION IN GENERAL

One thing that always troubled me, early on in my teaching career, was the fact that whatever assignments I used to give my students, I could never be sure that they understood the work they presented me in return. The fact that any particular student obtained a grade A or B was no guarantee that s/he actually understood the work clearly enough to be able to explain it to me in conversation.

Separately, I went through a personal experience relating to the writing of a set of course notes on the subject of Laplace transforms. It was in me having trouble finding a suitable analogy of what a transform was, and then resolving the issue by actually organizing my ideas, writing them down, reorganizing my ideas and rewriting them down, that I realized that it was in my attempt to interpret and reinterpret the subject that I actually learnt about it and how to present it.

By this experience it became clear to me why I wanted to include an element of interpretation in any work the students presented. In deciding to adopt an interpretative approach to the assignments I would give my students, I felt I would be able to see more clearly the degree to which they would illustrate their understanding. Also, their attempts at having to explain themselves in detail would provide the opportunity for deeper learning of the subject.

Since then my ultimate aim has always been to get students to interpret the text and mathematics of what they read. Consequently, in respect to a text being read, I mean *interpretation* to be:

a personally significant and valid re-description of the original text, based on the ever more refined and cultivated meaning you image of that text.

I shall therefore adopt this as the working description of "interpretation." An example of a text which a student would have to interpret in order to understand

and learn from could therefore be something like:

In elementary combinatorics it is shown that the number of partitions P(n,m) of a natural number n into m (not necessarily distinct) summands can be calculated with the recursion formula...

(G. Walther 1986), or

Confidence intervals and hypothesis tests based on large samples $(n \geq 30)$, discussed previously, rely on the fact that the statistic

$$\frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$

has approximately a standard normal distribution when $n \geq 30$. This follows from the Central Limit Theorem discussed in section 7.7.

(Chase and Bown p. 358, 1997). What generally tends to blind students about these types of definitions and expressions is the technicality/density of the language. To overcome this problem of non-understanding based on the type of language, students might then want to interpret these technical definitions *in plain English*, and more specifically in their *own* plain English, in order to develop a level of understanding of the technical language.

It would then be in the act of attempting to interpret the above definitions that students would be in a process of constructing a meaning to them. Given that their initial interpretations would probably be vague, incoherent and incorrect in parts, they could then go about refining these into more coherent, precise and correct descriptions.

What should be borne in mind here is not that I am advocating the simplification of the language of mathematics but that I am advocating its simplification as a means to developing a learning of mathematics, with continued interpretation as the process for that learning. Once the student has learnt to interpret and read the text, s/he will naturally talk about the subject at the more rarefied level of communication that more experienced mathematicians take for granted. The advantage from the student's perspective, however, is that s/he will now do so from a much stronger and personally more meaningful basis.

But, in order to do this, students need to learn how to read in order to use their reading as a basis for their learning. This implies that they need to use certain techniques for reading. However, beyond the mere use of techniques lies the domain of "strategy." Students need to be able to organize the way they use the techniques when reading-to-learn, depending upon the style of text they are reading (such an area lies beyond the scope of this article).

TECHNIQUES FOR INTERPRETING TEXT

The specific techniques which I have developed over the past three years can be classified into two families:

1) A family of techniques designed to help students interpret the detailed, micro level of the text that I call *KE**
2) A family of techniques designed to help students interpret the general, macro level of the text that I call Text Levels.

MICRO LEVEL TECHNIQUES: KE*

This family of techniques arose out of an experience I had with a student who had come to me for help with one of her subjects. Based on a set of notes her lecturer had given her, I proceeded to read a part of it and asked her if she understood a particular sentence relating to the definition of the word "stress" as relating to engineering. The sentence was:

The distribution of force across a section is called stress.

When she told me that she had not understood the sentence, I told her that the sentence didn't have to be written the way it was. I then showed her other ways the sentence could have been written; I did this by swapping parts of the sentence around to get:

Stress is the distribution of force across the section.

Another attempt at finding alternate ways of restructuring the sentence led me to:

Force across the section is called stress if it is distributed.

It is this last variation which led me to have an epiphany. Whereas previously I believed that I had completely understood the original definition, it was only with this last interpretation that I realized that I hadn't (well, not completely). It was in putting the word "distributed" at the end of the sentence that I finally understood the need for the force to be distributed. This is a point I had not been consciously aware of the necessity of. It then seemed that, in placing this word at the end, a greater emphasis was placed on it and allowed me to give a greater meaning to the term "stress."

I then came to name this technique of interpretation *Key Element Permutation* (KEP), whereby a person may choose any element of text as the key part to work with, and then permute them in any order. From this s/he would have to develop a grammatical and meaningful sentence around the new order of the elements.

> **...many students do not have a systematic and organized way of reading, and that is principally why they cannot learn from their reading.**

After this I developed other text manipulation techniques which, when used in combination with KEP, would provide students with ways of interpreting what they read. The two other main techniques were

Key Element Substitution (KES)
Here students swap chosen elements of text for either synonyms or elaborated explanations that they believe are most relevant. Such an approach to reading allows then to recast text in a personally more appropriate language, one which they understand fully. Such a language can then act as a point of departure in terms of refining their interpretation towards the level of language of the original text.

Key Element Deletion (KED)
Here students simply go about deleting parts of the text they feel are not necessary (what is defined as necessary or not tends to be discussed in conversation) to see if this helps focus in more clearly on the main theme of the text.

My experience in supporting student reading-to-learn suggests that KED is by far the simplest technique for them to put into practice from which useful meaning of text can be derived. KES is slightly more difficult to use for the purpose of interpretation since they are more unsure of what synonyms to use or how to elaborated upon the text. Indeed, their substitutions tend to be quite vague and imprecise (which is to be expected for text they do not fully understand).

KEP is by far the most difficult since they are never quite sure where to break the text up into elements or how to construct a new sentence around the way they have newly ordered the elements.

Now, you and I may recognize in these techniques aspects of the way in which we already read. The point is that many students do not have a systematic and organized way of reading, and that is principally why they cannot learn from their reading.

As an example, consider the G. Walther text presented earlier. Deleting certain elements and using synonyms for others one might develop an initial interpretation to be:

In (...) combinatorics *we can see* that the number of *splits* P(n,m) of a (...) number (...) can be calculated with the (...) formula ...

where those elements deleted are represented by the parentheses, and those element synonymized are represented in italics. Then, swapping parts of this sentence around may help to generate an alternative emphasis on it and therefore inform a new understanding to the student. Consequently, a student may develop:

A formula can be used to calculate the number of splits P(n,m) of a number.

As an *initial* interpretation this may be exactly the kind of description that the student understands. S/he may then develop a sequence of ever more refined interpretation leading towards the original text, but this time starting from a position of understanding and from a process of knowing how to interpret.

Macro Level Techniques: Text Levels (TLs)

This family of techniques, designed to help students focus in on the general gist of what they read, came to me during a class I was teaching on discrete math. I went into the very first lesson of the semester intending to guide students in their reading to learn using KE*. I had therefore decided to start the lesson by talking generally about a passage of the text we were using when it gradually dawned on me that I was interpreting the passage and doing so in an ever more generalized manner. I then realized that I was giving a one to two word descriptor to each of the particular paragraphs we were reading, and that in doing so it could be said that I was interpreting the gist of the paragraph.

In subsequent lessons I realised that I was adopting the same approach of "gist" interpreting the text, but this time for groups of sentences and then for individual sentences themselves. From this I thought of a hierarchy of "gist" interpretations based on the level of text, these being paragraph level, "groups of sentences" level, sentence level, etc...

Hence the idea of *Text Levels* (TLs) came to mind. They then have as their aim to allow students to elicit a meaning to the passage they are reading by initially *guiding* them into seeing whatever general idea(s) of the text they can. This guidance is given by the asking of an appropriate type of question such as "What is the text an illustration of?" or "What is the passage an example of?" The questions can then be altered to focus on whatever level of text the teacher may wish to guide their learning in.

As an example of the use of TLs consider the paragraph by Chase and Bown presented earlier. In order to focus the student's mind towards a particular level of text we might ask him/her, "What is each phrase of the sentence an illustration of?" from which a student may then interpret the text at the phrase level as:

for the 1st sentence
1st phrase: "Confidence intervals and hypothesis tests"
phrase interpretation: statement of techniques

2nd phrase: "based on large samples $(n \geq 30)$"
phrase interpretation: recap or summary

3rd phrase: $(\bar{x} - \mu)/(\sigma/\sqrt{n})$
phrase interpretation: formula

4th phrase: "has approximately a standard normal distribution when $n \geq 30$"
phrase interpretation: statement

for the 2nd sentence
1st phrase: "This follows from"
phrase interpretation: linking or justification comment

2nd phrase: "the Central Limit Theorem discussed in section 7.7."
phrase interpretation: naming of theorem

From this we can see that concentrating on the phrase level of text should help focus the student into constructing a more specific meaning to the text.

Beyond merely interpreting the gist of a particular level of text, there remains the fact that they need to be able to interpret the chunk of text as a "whole." Having interpreted the gist of the paragraph phrase by phrase, the aim now would be for them to be able to synthesize these descriptions into a coherent summary. This is done simply by creating a sentence or two out of the separate text level descriptions in order to develop a *Text Level Construction.* For example:

"This paragraph talks about some techniques and then makes a summary. It then shows a formula and makes a statement, and finishes with a statement and names a theorem."

With this interpretation in mind, the student may then compare with the original text to personally judge the viability of his/her understanding of the original text.

Now, the importance of constructing a sentence from previously separate text level descriptions should not be underestimated as a learning opportunity. By the process of creating a TLC the student now has the opportunity to develop the ability to express, clearly and coherently, the meaning s/he has generated from the text, and do so from his/her level of language. In doing so it makes explicit the extent to which the student has been able to construct a whole-meaning to the text.

The whole purpose of these techniques is to allow students the ability to develop a fairly detailed and precise meaning for the text they are reading. Consequently, a student would aim to develop a final interpretation such as:

"This paragraph talks about how two types of statistical analyses, which use a particular formula, are based on a specific requirement. A particular theorem is then stated as justifying the validity of this requirement."

It would be a trivial step for the student to then identify the particular types of statistical analyses (i.e. confidence intervals and hypothesis testing), the particular formula, the specific requirement (i.e. normality) and the actual theorem used as justification.

HOW IT ALL WENT

All in all, having settled down in my own learning of how to guide student learning, the vast majority of students came away with a much improved ability at learning mathematics (certainly all of them thought that, at least, this was a useful experience to go through even if they did not intend to carry on with this approach to learning the subject).

They were able to read into math a meaning they had not previously seen or even thought they would be able to see. As a consequence of this, their personal attitudes towards the subject of mathematics itself were considerably changed for the better. So, not only has their level of math improved, but they can now see how they themselves might go about improving their level of understanding beyond that they developed in the sessions we had.

Furthermore, some of the students actually went beyond merely using this approach in their sessions with me in that they automatically went about looking, thinking about and learning from what they were reading in other modules of their courses, *without me ever having suggested that they do this*.

An example of the usefulness the techniques have had in supporting students' attitudes towards the subject and their learning of it can be seen in the excerpt of conversation below, which I held with some of my students at the end of the course. The three participants of the conversation below are myself ("C") and three students N, M and K.

C: How did [the module] compare to normal math modules that you might have done in the past?

K: At first when I first started the module I was thinking, "Well, is this math or English?" But then I understood why you were doing it because I was starting to understand the subject and pick it up faster.

N: Yours [the lecturer's] was a better method.

K: [...] your approach was so much different, so much easier. I mean, I can read anything now, that I might not have been able to before in math.

M: I thought it was excellent. Yeh. It just let you think about math in a different light, a different way. I found it making math so much easier. Yeh, I used it elsewhere. Because in some of my lectures you've got loads of information. [...] I even use the techniques for some of my exams because I just basically cross out some jargon. [...] So I thought it was excellent.

C: [...] Nimisha, any comments?

N: Well, I thought it was pretty good the way you taught us. I got to understand the subject more. The technique was really good because I could apply it, and I could understand the actual math. Before I couldn't do that.

C: [...] Right. So let me ask you again what aspects did you think you liked best in terms of content, the lessons themselves, the text reading stuff, anything, etc...

N: The text reading. I thought that was really good, and it involved the actual student. [...]

C: And has your view about your like or dislike about math changed?

N: Yeh, I like math now. (*Laughs.*) I never used to like it that much because I couldn't understand it. But now I can understand it; I enjoy doing it.

M: Well, I've always loved math, but math was never my strong point. But with this technique I can now use it and understand math a bit further. So, I found it really, really, really useful. (*Laughs.*)

C: What kind of insight do you think this [approach to the course] has given you into ways of reading or ways of learning?

N: Well, now when I look at something I feel that I don't have to understand it straight away, and I have something there that I can use to help me understand that paragraph in time. So it's good.

C: [...] What experience do you feel you have gained [and] what do you value about the experience?

N: I've learnt how to read text, and I've started to understand math better than I did before.

C: Right, and what do you value about having that experience?

N: Getting to know math better and actually working with it. Because I know I need math for everyday life, so I'm so glad that I've actually gained that knowledge, that understanding of knowing what each bit is, and getting to know my math better.

From these comments it therefore seems that, when used as a classroom activity to support learning, students are able to approach their reading in a way they were not able to before, and consequently understand and learn mathematics beyond what they had thought possible.

WHAT OTHERS ARE DOING

A plethora of reading techniques abound in the reading research literature (see for example *Reading Research Quarterly* or the *Journal of Reading*), most of which are based on developing students' abilities at comprehending the gist level of text. Schwartz (1980) tested the different demands required of readers to comprehend text at three different levels, namely whole text level, individual word level and letter level. Previous work (Meyer, Brandt, Bluth (1980), Rinehart *et al*. (1986), Richgels *et al.* (1987)) suggests that good readers, those able to identify and follow a text's major themes and relationships as well as the facts supporting these themes, use a structure strategy when reading, but that poor readers lack precisely this skill. Consequently, Meyer, Brandt, Bluth (1980) developed a structure strategy which was designed to follow the organization of the author's text structure and allow students to focus on finding connections between large chunks of the text they were reading, while Rinehart *et al*. (1986) studied students' abilities at summarizing what they read by getting them to identify and delete certain types of information, as well as relating the main ideas they found to relevant supporting facts.

Little has been done in terms of helping students read-to-learn at the micro level of text. However, some work in the area of manipulation-type techniques includes that of Straw and Schreiner (1982) who developed sentence-combining and sentence-reduction techniques for helping students better understand the text they were reading. Ross (1972) has concentrated on sentence manipulation and transformation techniques (although not in connection with reading comprehension) while Rinehart *et al.* (1986) and Brown, Campione and Day (1981) studied students' abilities at understanding texts by using, in part, the deletion of certain types of information in order to summarize the major and minor aspects of what they read. Bean and Steenwyk (1984) have also used elements of deletion, as well as substitution (based on the work of McNeil and Donnant (1982)), in order to improve students' summaries of the texts they were reading.

Weiss (1983) argues that the reason poor reading comprehenders are not poor listening comprehenders is that oral discourse is marked by (amongst other aspects) pauses which are not marked in written discourse. His rationale was therefore that students could develop into better comprehenders if such pauses were introduced into the text, this being done by segmenting its sentences. Weiss then tested two types of segmentation based on spacing out the phrases of a sentence according to either their grammatical/syntactic structure (of noun/verb phrases, or compound phrases which framed a particular idea) or their pausal structure (this being defined as a unit of utterance between two pauses of breath which would occur during the speaking of a particular sentence).

Other text manipulation techniques developed to improve reading comprehension include those of Weaver (1979) (*RRQ* 15(l)) who developed a way to solve sentence anagrams by using a word grouping strategy. Her aim was to improve reading comprehension by teaching students how to go about organizing their construction of sentences. Consequently, the sentence anagram technique required students to form a sentence out of a jumbled set of between five and fifteen words, each word having been written on separate cards. Students were then taught to construct their sentences by attending to the general structure of language. Consequently, they grouped words into phrases and grouped phrases into sentences, thus providing a structural approach to the construction of their sentences (the students were taught to construct the phrases themselves by using a verb as the focus and "enframing" it by action words).

In fact, very little work has actually been done in reading-to-learn in mathematics. The two principal people involved in this area are Raffaella Borasi and Marjorie Seigel (Borasi *et al*. (1998), Seigel and Borasi (1992), Borasi and Seigel (1990), Seigel and Fonzi (1995)). The majority of these studies focus on the reading of narrative style texts and are based on an approach to reading known as transactional theory of reading which involves the reader in actively participating with the text s/he is reading. Consequently, the reader is supported in a new way of thinking and engaging with a text by certain techniques which aim to foster a generative approach to interpreting and learning from it. Four approaches these workers have developed include *Say Something* (a type of free association technique), *Cloning an Author* (which involves developing a map of the interrelationships between what the reader considers to be the important ideas of the text), *Sketch-to-Stretch* (in which the reader sketches an interpretation to the text) and *Enactment* (in which the reader aims to act out the story of the text).

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Journal Review: *Third International Anthology on Paradoxism*

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The Third International Anthology on Paradoxism is available from Bell & Howell, 300 N. Zeeb Road, P. 0. Box 136, Ann Arbor, Ml 48106-1346 or http://www.umi.com/bod/

Recently I received a copy of this fascinating journal of paradoxist, tautologic and dualistic distichs by writers from fifteen nations, including the United States. You may be unfamiliar with the "distich" but, before I give its definition, it is pertinent to describe the movement out of which this term emerged.

A new movement in literature called paradoxism—which makes heavy use of opposites (antitheses, contradictions, oxymorons and paradoxes) at both local and global levels in creative work—began in the 1980s in Romania. Its initial driving force was an anti-totalitarian protest against a closed society where the entire culture was manipulated by a small group. Anthology editor, Florentin Smarandache, who now lives and teaches in New Mexico, responded to the crisis with the idea, "Let's do literature...without doing literature!" In short, by keeping silent and, for example, observing that a bird in flight is itself a poem (a "natural" poem, needing no words).

This beginning then led to an emphasis on contradictions. Most in Romania lived a double life—an official one conforming to the political system and another "real" life. People said "life is wonderful" when, in reality, "life is miserable." Language opposites were flourishing! Thus paradoxism was born. Folk jokes, which said one thing and meant the opposite, were very prevalent during Ceausescu's era.

Paradoxism has introduced a number of literary terms; here are several of them:

Paradoxist distich: a two-line poem in which the second line contradicts the first, but both lines together form a sensible explanation of the title.

Example (by Smarandache):

SCAPEGOAT
Even if he didn't
he did

Tautological distich: a two-line poem that appears to be redundant, but the pair of lines deepens the explanation of the title.

Example (Smarandache):

IMITATOR
Discovered
What others have already discovered

Dualist distich: a two-line poem in which the second line is the dual of the first, and together they explain the title.

Example (Smarandache):

MULTIDISCIPLINARY
History or art
Or the art of history

The Third International Anthology on Paradoxism entertains and puzzles its readers with nearly one hundred pages of distiches and variants, all offered in English but some also provided in Chinese, Italian, Romanian and Spanish.

Here are several samples:

from Paul Haugh (Australia):

CUTTING REMARKS
Sharp as a knife
Blunt as a cork

from Paulo Bauler (Brazil):

ORDER
Someone with all the reasons is
Somebody with no reason

from Maria do Carmo Gaspar De Oliveira (Brazil):

DISCOVERERS
Portuguese discovered Brazil
Already discovered by Indians

from Victor Chnagnone (China):

ENEMY
Fails
When we succeed

from Richard Cheevers (England):

URBAN JUNGLE
On a London street
Zebra crossing

from Anand Rose (India):

WISE
When you know
you don't know the answer

from John Grey (USA):

MAD WORLD
you'd be crazy
to be sane

Real Numbers, Math Lives

Arnold Trindade
Glen Cove, NY

It was so true the cone did age
Ten-fifteen billion
It is so true the earthly age
Four and a half billion.

It is yet true my only age
The forties
My end, it will surely come
The eighties-nineties.

Nature, her organic-inorganic wonderings
In the wings, on cue
Appear, disappear
On a temporary reflector revolving.

See trembling aspen leaves, a grouse
Alluring nest on the ground hatches
In twenty-one days the fledglings drowse
From on high the goshawk watches.

See this year the drumming grouse
Their numbers few, heard scarcely
The goshawk lays eggs only two
Instead of four when prey a plenty.

See cold Arctic the hunting lynx
Pursues the snowshow hare, a meal
See as sunlight the flowerings control
So hare fleets the lynx withhold.

The numbers when genius Jason dawned
Bursting forth at nine and two
See poor Jimmy come faster born
Came four weeks with raw limbs too.

Earth a scientific developer
Counts hours, days, seconds too
Releasing light darkness covers
In revolutions and leap years new.

Can we figures, give up forget?
Can we cycles, senses insensate?
Do we deny light-matter nets
Food webs, equilibria, numbers-kind?

Thus the dispersing universal cone
A developing matrix electromagnetic
Her offspring, her coordinates dynamic
Show moving graphs, patterns, bones.

Thus do light packets, photons
Imprisoned energy, nucleons
In seconds, minutes exposed
Produce on matter-film sequential codons
Alluring pictures, images
The living kind!