What We Say, What Our Students Hear: A Case for Active Listening

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I want us to think about what our students hear, which is often not what we are trying to convey. I think we can all believe that things go on in our students' heads that we don't understand and that we need to pay more attention to. I suspect that more than one of you has put a problem on a test that lots of students have answered incorrectly and you've said, "How could they mess that up?" I hope that this paper will give you some clues about why they "messed that up."

More importantly, I want to encourage you (and me) to listen more carefully to what our students do say. The "active listening" in this paper involves listening on OUR parts.

MESSAGES OUR STUDENTS HEAR

What are some messages that our students hear?

We say, "This won't be on the exam."
They hear, "This is not important."

We say, "You will need this concept next year."
They hear, "I don't need to learn this concept this year."

We say, "We want you to use algorithms quickly and automatically."
They hear, "Mathematics does not require thought."

We give timed tests.
They hear, "Mathematics must be done quickly." They, therefore, will not struggle with problems that they cannot complete quickly.

We give them lots of exercises with no words.
They hear, "Mathematics is not a language of communication, only computation."

We don't give partial credit.
They hear, "The mathematics is the final result, not the process. Mathematics is either all right or all wrong; there is no middle ground."

ONE MODE OF REASONING

To begin to understand more deeply what our students hear, I want to think about mathematics, and about one suggested style of reasoning that people might use in mathematics.

- Gets right to solution in a structured, algorithrmic way, stripping away any context.
- Uses a mode of thinking that is abstract and formal.
- Geared to arriving at an objectively fair or just solution upon which all rational persons can agree.
- Employs a legal elaboration of rules and fair procedures.
- Confident to judge.
- Is analytic.

How does this reasoning style relate to mathematics? Think for a moment about this reasoning style. Does it describe mathematics for you? Do you think it describes mathematics for your students? Is it like reasoning in mathematics? What are its limitations?

Is this what we would like our students to be able to do with the mathematics we teach them? Let's think about this reasoning mode.

Gets right to solution in a structured, algorithmic way, stripping away any context.
We want them to be efficient. We want them to be able to apply the mathematics to varied situations. We want them to use the precise, formal, logical processes that we have taught them.

Uses a mode of thinking that is abstract and formal.
Of course!

Geared to arriving at an objectively fair or just solution upon which all rational persons can agree.

Isn't this the goal of mathematics? Mathematics is well defined. When you pick up a mathematical object you know immediately whether or not it fits the definition at hand.

Employs a legal elaboration of rules and fair procedures.
Mathematics is a formal system. The definitions, axioms, theorems, algorithms, give us a common basis for discourse.

Confident to judge.
Because it is well defined, we can work within it with certainty.

Is analytic.
Mathematics is an analytic system, which we develop and use analytically.

> ***We say, "This won't be on the exam." They hear, "This is not important."***

But when we read this list we probably do not all hear the same things that our students hear. A mathematician may read "abstract and formal" and see an abstract system that is pretty well laid out. If there were any ambiguities, we have removed them by our choice of definitions and by staying in a small domain. It is complete with underpinnings of human exploration that we know are present.

But many of our students hear "abstract and formal" as "coming from outside without meaning." Many believe that they need to give up their own ways of thinking and memorize these algorithms, definitions and proofs that are meaningless to them. They do not see a system that is applicable to many contexts. They see isolated sets of instructions that are highly compartmentalized.

I have shared this reasoning style with many in mathematics and mathematics education over the years. Most have agreed that this list illustrates the way that mathematics is conveyed in the classroom, in traditional textbooks, and in our professional writing. We present elegant, well-polished proofs, carefully devised sets of examples, collections of theorems and corollaries, and sets of applications. Mathematics is polished and complete. (See Buerk 1985, pp. 63-64.)

But what we present is like the part of an iceberg that we see above the water. WE know what the tip of the iceberg is sitting on. We may know exactly what the underpinnings of a theory are. If not, we do know that what we see has underpinnings that we could study if we wanted to.

Many of our students see just the tip of the iceberg, which we present to them. They have no idea what is under the surface of the water. They may describe what is under the water in a very different way than we would. Some even believe that the theory or concept has no underpinnings, that it has no human or mathematical connections.

We want our students to know that mathematics is grounded in human thought, human exploration, and human questions. Will this happen if we only present them with the tip of the iceberg? Will this happen if we only share with them the public image of mathematics as a completed formal system?

STUDENT VOICES
Let's listen to several articulate students who have seen or heard only the public image of mathematics-only the tip of the iceberg.

> Peg writes:
> On the eighth day, God created mathematics. He took stainless steel, and he rolled it out thin, and he made it into a fence, forty cubits high, and infinite cubits long. And on this fence, in fair capitals, he did print rules, theorems, axioms, and pointed reminders. "Invert and multiply." "The square on the hypotenuse is three decibels louder than one hand clapping." "Always do what's in parentheses first." And when he was finished, he said, "On one side of this fence will reside those who are good at math. And on the other will remain those who are bad at math, and woe unto them, for they shall weep and gnash their teeth."
>
> Math does make me think of a stainless steel wall - hard, cold, smooth, offering no handhold, all it does is glint back at me. Edge up to it, put your nose against it, it doesn't take your shape, it doesn't have any smell, all it

does is make your nose cold. I like the shine of it—it does look smart, in an icy way. But I resent its cold impenetrability, its supercilious glare. (Buerk 1982, p. 19)

Here is a creative, insightful woman who presents a view of mathematics as created by God, not by human thought. Mathematics is a fence separating people—for Peg it separates those that are good at mathematics from those that are bad at mathematics. That division is absolute, for the fence is too high and slick to climb over, and it is too long to go around. The fence presents to us all the rules of mathematics. Mathematics and this stainless steel wall have no human warmth, no smell, no flexibility—just a "cold impenetrability" and a "supercilious glare." This view of mathematics as an absolute, closed system with no human connections is clear and well defined for Peg. She would like a way to connect with mathematics, but finds none.

Jackie, a second student, writes:
I was exposed only to the public image of mathematics. To me, there seemed no room for interaction with the content, no possibility of connection with the ideas. Mine was the role of tourist who merely looks out at the sights that surround [her] as they travel past in a blurred rush. (Buerk & Szablewski 1993, p. 151)

Jackie, like Peg, wants to find a way to connect with mathematics. She feels like a tourist on a whirlwind tour with no time to catch her breath or appreciate the sights. She elaborates, in an assignment to write a letter to her next mathematics teacher:

I realize that in order to help us realize all that already exists in the world, in order to guide us through all the worlds of mathematics, you must keep to a strict itinerary. If you didn't, we would not be exposed to all we must be exposed to in order to reach the destination of "mathematician," "chemist," "well-rounded person." But don't you see that in your well-intended efforts to show us all the "landmarks" of those worlds, you are not allowing us to touch? How can we come to say that we believe in a thing, a concept, an idea, if we ourselves do not know it is real? (Buerk & Szablewski 1993, p. 152)

Jackie lets us know how frustrating it is to not be allowed to touch and experience mathematics. She also shared her discomfort in the mathematics classroom in the following:

Unlike English class, math was not a place for ideas in process. You could not say or share something you were thinking about. You could only share with the class completed perfected thoughts, and I simply had no such thoughts concerning math. (Buerk & Szablewski 1993, p. 152)

For Jackie, mathematics requires a kind of thinking different from her own, because her thoughts in mathematics do not come out as completed thoughts. She needs to slowly develop her thoughts, but she believes that her own thoughts are not allowed in the mathematics classroom. Since she believes that she cannot think in mathematics in a way that works for her, she becomes silent. Seeing only the tip of the mathematical iceberg reinforces these beliefs in many of our students, even when we tell them that mathematics is more than what they see on the surface.

Jackie tells her story in an article we wrote together in MAA Notes #32, *Essays in Humanistic Mathematics*. The essay is entitled, "Getting Beneath the Mask, Moving Out of Silence."

A third student, Lee, wrote,
Doing math can result in a precise answer or an estimate but it is not a thinking process. Rather it is a process of identifying, comparing, and doing a problem in relationship to that identification and comparison. (Buerk 1990, p. 80)

Be careful as you read this. Lee means that he does not think when doing mathematics. His process is mechanical. For him "identifying and comparing" mean that he tries to find the correct algorithm or procedure.

Listen to Lee explain:
The math process is one in which all attention is focused on a narrow subject. My mind is not allowed to create, or wander, or to think

> about doing the problem. My mind says, "Compare this problem to others that are like it and base your answer on the way you found your answer to that other problem." (Buerk 1990, p. 81)

If there is no algorithm in his collection then he cannot solve the problem, he has no recourse, but to give up and wait for someone else to solve the problem for him.

Peg, Jackie, and Lee are showing us a conception of mathematics that is quite different from our own. For them mathematics is a complete, closed system. While it is important, it is beyond their grasp. It seems very cold; it is not something they can relate to or touch. They do not believe that mathematics was created by people. They believe that learning mathematics requires following exact patterns, algorithms, and rules that someone else has given to them. They view mathematics as rote. For many students holding this rote conception of mathematics, doing mathematics means putting aside their own thinking, and instead, memorizing algorithms that have no meaning for them.

STUDENT INTERACTIONS

Let's look at some examples of interactions with students who have rote conceptions of mathematical knowledge.

First let me tell you about my encounter with Jake, a college freshman. Jake accepted a rote conception of mathematics without concern.

> In a class discussion about exponents, [he] told me that exponents were added when multiplying factors with the same base. I asked him why. He said, "That's the rule." I asked him why the rule said that. "It just does," he replied. "It is the rule I was taught." "But why?" I persisted. He looked at me very seriously and asked, "You mean there's a reason?" Jake was very surprised to hear that there might be a rationale for this mechanical manipulation. (Buerk 1985, p. 60-61)

Second, let me share with you a situation that is not atypical. I'm sure that we all work to justify as many algorithms, extension of rules, and definitions as possible. In that spirit, consider the following:

Said at the blackboard:
Question: If we decide to rewrite $\sqrt{x}$ in exponential form, x^r, why does the value of r have to be 1/2?
Suppose we assume that there is a real number r such that $x^r=\sqrt{x}$
We know that: $\sqrt{x}\,\sqrt{x}=x$.
Since $x^r=\sqrt{x}$, we can replace $\sqrt{x}$ by x^r, so $x^rx^r=x$.
Now we want to preserve the rule (pattern) $x^ax^b=x^{a+b}$, that we know works for integers, so we can write $x^rx^r=x^{r+r}$.
Then $x^{r+r}=x$.
Therefore, $x^{2r}=x$.
Which tells us that $2r=1$.
Hence, $r=1/2$and, therefore, $\sqrt{x}=x^{1/2}$.

We go through an explanation something like the one above, and we write the steps on the board as we go along.

Written on the blackboard:
Question: If we decide to rewrite $\sqrt{x}$ in exponential form, x^r, why does the value of r have to be 1/2?
Suppose that $x^r=\sqrt{x}$.
$\sqrt{x}\,\sqrt{x}=x$.
$x^rx^r=x$.
$x^{2r}=x$.
$2r=1$.
$r=1/2$

My colleague, Ann Oaks, the Chair of the Mathematics and Computer Science Department at Hobart and William Smith Colleges in Geneva, New York, saw a student in her office the day after she gave an explanation like the one above. She happened to notice his notes. He had written the following:

Student Response:
Step 1: First you set $x^r=\sqrt{x}$.
Step 2: Then you set $\sqrt{x}\,\sqrt{x}=x$.
Step 3: Then you set $x^rx^r=x$.
Step 4: Next you write $x^{2r}=x$.
Step 5: Then you say $2r=1$.
Step 6: $r=1/2$

Not only had the student turned the explanation into a set of steps, he did not even include the question his steps were responding to. Ann checked with her class and found that many others had the same kind of notes. Her explanation had become another procedure to be memorized.

Many of our students are not listening to our explanations as a way to help them make meaning of the concept for themselves—to really understand the concept. They do not see that we are trying to place the definition in a context for them. No, the students are seeing our explanation as another procedure for them to learn.

> **"**
> ***We say, "You will need this concept next year." They hear, "I don't need to learn this concept this year."***

Third, in a recent research project at a small liberal arts college, two good Calculus II students were given two problems to solve, and their process was audio-taped. The first problem was a rather easy integral, but the process to solve it was not obvious. Students had to stick with it, but they had the skills to solve it. The second problem was a mathematical puzzle problem that was much harder to solve than the integral problem. The students gave up on the integral fairly quickly, because their instructor had not shown them how to do that type of problem. Their rote conception of mathematics took hold. But, they stayed with the puzzle problem until they got the solution. They kept saying, "We can keep trying." They solved it. They saw the integral as a classroom problem. They gave up because they had not yet been shown the algorithm to use. Like Lee, they believed that they needed an algorithm, or they could not approach the problem. The other problem was not a "school" problem, so they used their ingenuity to solve it. For other problems, non-school problems, they could explore. Their rote conception of mathematics applied only to "school" problems.

While many students' view of "school" mathematics problems is disturbing, it is exciting to notice that our students still retain their innate intellectual curiosity. We can, and must, build on this curiosity. We must couch more of our problems in forms that prevent our students from looking for past models and, instead, encourage their creative thought. We can ask questions differently. Many students have a mechanical method to approach "solve." We could try "verify," "explain," "explore," "discuss," or "describe to someone not in this class." With these options many begin to realize that they are being given permission to think.

We must be careful when we listen to students. They may use words that give us hope that they are really understanding the mathematics that we are teaching them, when they are not. For them "math is important" may refer only to the kinds of questions we ask on tests and quizzes. For them, "math is useful" might refer only to the kinds of mathematics problems that they can solve. When they focus on "understanding" they often mean that they know which algorithm to use. We must listen carefully, or we will not really hear how our students view mathematics. And sometimes their views are ones that we really do not want to hear.

The reasoning mode we have considered conveys to many only the public image of mathematics, only the tip of the mathematical iceberg. But if our students have a rote conception of mathematics, they do not use our first reasoning style. In fact, they do not reason at all.

A SECOND REASONING MODE

Let's look at a second suggested style of reasoning.

- Tries to experience the problem, relate it to personal world, clarify language, create context, remove ambiguity.
- Uses mode of thinking that is contextual and narrative.
- Geared to looking at limitations of any particular solution and describing the conflicts that remain.
- Tolerant in attitude toward rules and more willing to make exception.
- Reluctant to judge.
- Is intuitive.

How does this reasoning style relate to mathematics? Think about this reasoning style for a moment. Does it describe mathematics for you? Does it describe mathematics for your students? Is it like reasoning in mathematics? What are its limitations?

My colleagues with whom I have shared this reasoning style find that it relates to mathematics as well. Their consensus is that this reasoning style represents the way that mathematicians do mathematics. "Mathematics is intuitive," they say. They stress the creative side: attention to the limitations and exceptions to theories, the connections between ideas, and the search for differences among theories and patterns that appear similar. (See Buerk 1985, pp. 63-64.)

Think about this. Mathematicians use a reflective, contextual, groping strategy to develop mathematics, but they share with the world ONLY the polished, finished product, which gives no clue to the process used to create it. Mathematics has a public image of an elegant, polished finished product that hides its human roots. It has a private life of human joy, struggle, challenge, puzzlement, and excitement. This works well for us, who know both the private and public worlds of mathematics. It does not work well for the student who sees only the tip of the mathematical iceberg and does not know something is below the water's surface.

For me, personally, mathematics is creative, dynamic, and evolving. I value its personal, intuitive, logical, and reflective dimensions. I enjoy the process through which mathematics is created. This process, which I see as involving conscious work, unconscious work, intuition, conjecture, reflection, redefining the question, asking new questions, and finally a degree of certainty, is a very human one resulting in a formal, logical, and consistent presentation of a complete idea. As you finish one problem or proof you are often left with many new questions to pursue. This process is used by mathematicians, educators, students, me, and in fact, by any inquisitive person approaching ANY question that is new to them. I want my students to know this side of mathematics as well.

> **"**
> ***We say, "We want you to use algorithms quickly and automatically."***
> ***They hear, "Mathematics does not require thought."***

By accepting the public image of mathematics, many thoughtful people find our discipline easy to reject, for it seems not to offer the opportunity for their own thought. Others find this image intimidating; they struggle to model someone else's thought process without truly understanding that process. Others reject mathematics because they find no way to include within it their own intuitive understanding of quantitative concepts or to use it as an opportunity to further their own quantitative intuition. Our presentation of mathematics in traditional ways gives a distorted picture of mathematics to many who are not in our discipline. Peg, Jackie, and Lee have made this clear.

STUDENT VIEWS OF MATHEMATICS

Our students form views of mathematics based on what they hear about mathematics in our culture, but more importantly, what they hear and experience in their mathematics classes. Let's listen to some more views of mathematics held by some mathematics students. Think about how each of the following students views mathematics as you read his or her words.

> First,
> Math is most like an earthquake. If an earthquake was to hit, even just a tremor, it could knock down and ruin a lot of things. Just like in math, if you make one error in a problem, even a small one, it can ruin or tear down all of your work. (Gibson 1994, p. 8)

This student is saying math is something over which he has no control. In fact, mathematics is like a natural disaster over which no one has control. Also only the correct answer matters. In his classroom he gets no credit for the process or for the ideas that would lead to the correct answer. He does not have confidence in his ability to work with mathematics. He is not in control.

> Second,
> Doing math is like driving through a city that you used to know but that has grown more complicated in your absence. You start down certain streets that seem familiar, but then you realize that these are dead ends. You can see the place you're trying to get to, but the way is full of detours and traffic lights that seem to be stuck on red...When you finally get to your destination, it feels good that you've traversed this dangerous and confusing city, but sometimes the thrill isn't worth the effort put forth

> to get there...Math is scary, like a big city. (Buerk 1996, p. 28)

For this student doing mathematics feels like being subjected to unnecessary roadblocks that often make the effort to maneuver through them feel like time wasted. While this student may be successful in mathematics, she finds much of what she learns to be a waste of time. She knows that she will not be able to retrace her steps the next time she comes to that city. She will probably take only the courses that are required for her major.

> A third,
> For me math is like a toolbox. The tools in the toolbox represent concepts, formulas, and techniques needed to solve problems. However, I could always use the wrong tool, or maybe my toolbox doesn't have a tool I need. The tools can be used to construct something, or they can be used to strip down a complicated machine so that all the parts can be analyzed. Some tools can become obsolete if I acquire new ones. When working with the tools of mathematics, I could just as easily use them to fulfil my needs by solving problems. (Buerk 1988)

This student, unlike the first two, feels empowered and reasonably confident in his ability and skills. He does the mathematics asked of him. We must hope that his tools are not just algorithms and procedures, but that he has other tools that are reasoning skills and problem solving strategies. The use of a tool analogy could be very limiting. Does the student see various uses for each tool? Can the student create his own, new uses for a tool in his toolbox. We would like this student to have the tools to work with mathematics in an integrated, not compartmentalized way.

> And, a fourth,
> To me, math is like a used car that you get for a good price: sometimes it runs smoothly, but on certain days things go wrong. It's frustrating, like a car can be, when it won't go right. You have to sweat, yell and curse, and sometimes pay a price to get the car going, but once it does go everything's great. With math, things don't always work out right. I don't know how many times I've screamed and pulled my hair out trying to "fix" a math problem, but when I finally figure it out, I feel fantastic; like I've accomplished something. Sometimes you break down, in a car or during a math problem, but if you work with it, you'll get to where you've going! (Gibson 1994, p. 9)

This student finds much in mathematics to be a struggle, but she stays with it. She doesn't give up easily; she has to work for her successes. She knows she is travelling in an old car, lacking many of the advantages of a newer one. Therefore, she knows she must compensate for having the old car, by developing a determination to fix things that go wrong. The determination we hear from this student is missing from many of our students who give up much too easily and wait for someone else to give them the answer.

I hope that you are beginning to hear how your students view mathematics. Their rote conception is deeply imbedded in their beliefs, and, therefore, very difficult to change. While students may not like mathematics as they view it, it is the only mathematics they know. Remember the example I gave about the square root of two? The instructor was trying to help her students understand that their definition of fractional exponents was consistent with the system that they were using. The students saw this explanation only as another procedure to follow. As we try to help our students see mathematics as a human endeavor, we will face resistance.

Ann Oaks, Hobart and William Smith Colleges, teaches a course called "Discovering in Mathematics," in which students actively create mathematical ideas and have lively discussions about them. They do become excited and do approach mathematics in very different ways. However, her students consistently call the course, "Discoveries in Mathematics." They do not see themselves discovering mathematics. They do not see anyone discovering mathematics. They still see themselves as learning about the discoveries that are mathematics. While they can change their behavior, it will take more experiences to fully change their rote conception of mathematics.

OUR TWO REASONING MODES

I want to come back to the two suggested reasoning styles that we have discussed. They may sound a bit awkward to you. They were not originally written with mathematics in mind. I drew them from another source and asked my colleagues to help me to understand how they might relate to mathematics. It was my colleagues who helped me see the connection to the public and private worlds of mathematics. It is clear that mathematicians need a blend of these two styles. One to create and do mathematics, and the other to present to the world the mathematics that we develop and use. We could not do mathematics if we did not integrate these two methods of reasoning.

I developed these lists back in 1983 while reading the work of Carol Gilligan, especially her book, *In a Different Voice*, and the related work of Nona Lyons. In dealing with hypothetical and real moral dilemmas they found that men tended to prefer the first reasoning mode, which is often called "separate" and that women tended to prefer the second reasoning mode, which is often called "connected."

> "
> ***We give timed tests.***
> ***They hear, "Mathematics must be done quickly."***

By sharing only the tip of the mathematical iceberg, the public image of mathematics we are encouraging a rote conception of mathematics. We are encouraging students to memorize symbol strings that they find meaningless. But, by sharing only the tip of the mathematical iceberg, the public image of mathematics, we are also reinforcing to many females the cultural stereotype that mathematics is a male domain.

It is exciting today that because of the reform movement in mathematics education, we have a wealth of pedagogical and curricular resources available to help us show students more than just the tip of mathematical iceberg. Materials from the various reform projects encourage students to use both of these reasoning modes. Students are given experiences in the second mode to explore, to experience, to follow their own thoughts in process, to listen to the ideas of others, to touch, to feel, to work collaboratively, and to write. These experiences are necessary to help break down rote conceptions of mathematics and to see what is supporting the tip of the iceberg. The students can then see, and even help define, the finished product, and see it with meaning.

It is my personal hope that as we incorporate more of the ideas of the reform movement at all educational levels, we will see fewer rote conceptions of mathematics and less fear of mathematics in our students in the future.

I am of the generation that saw and participated in remarkable changes in opportunities for females. In the fifties my interest in mathematics was encouraged, but my only career option was teacher. While I never regretted that career choice for me, I have worked very hard in the ensuing years on issues of gender equity in mathematics. I have also worked hard to have the strategies of the reform movement included in mathematics classrooms. Those of us using these strategies in the seventies thought of them as feminist pedagogy, though we avoided using those words as much as possible. It has been exciting for us to see them come into the mainstream in the 1989 Curriculum and Evaluation Standards for School Mathematics and in the reform movements in mathematics education.

A MATHEMATICIAN'S STRUGGLE TO HEAR HIS STUDENTS

I have tried to help you understand what our students hear and to some extent why they hear it. I hope that I have helped you to see why the strategies of the reform curriculum are important for our teaching. But the fact remains that listening to our students can be difficult. We really have to practice hearing what they are really saying to us. To reinforce this idea I have chosen the words of David Henderson, a research mathematician at Cornell University, who describes his experience eloquently in "Mathematics and Liberation," which appeared in *For the Learning of Mathematics* in 1981. He says:

> Let me relate what happened to me when I started teaching calculus for the first time (after I was already an established mathematician).
>
> I tried to listen to the people in the class. I tried to understand what their questions were. I found that some people were not thinking clearly because of emotional problems or because of rigid reactions that came from previ-

> ous conditionings. But other people were obviously thinking clearly, and I tried to understand what they were saying. In many cases I found this terribly difficult—my gut reaction was that it couldn't possibly be right—it felt like nonsense. I felt threatened—here was something which I couldn't see in an area I felt certain about.
>
> Gradually, after much persistence and with the help of friends, I began to sense that I had blinders on—that my ways of understanding calculus had blinded me to other ways of perceiving. I saw that many of the people in the class had real questions about the meaning of limits and derivatives—questions which I could not answer or questions which I then started to explore for the first time. I lost a certain narrow feeling of certainty but gained a broader perspective. Now I perceived calculus in a different way. (Henderson 1981, p. 12)

Henderson documents his struggle to learn to really hear his students. In his essay he then reflects on what his lack of hearing, lack of listening, might have meant to his students.

> What was happening to the people in my class who were asking a real question I couldn't understand? Some correctly sized up the situation and blamed my blinders, but this was rare. Most blamed themselves.
>
> It is a hurtful experience to have someone whom you see as an authority not understand a real question of yours. When this and other distressful mathematics experiences happen to people enough times over the years, they feel stupid, they feel they can't think about mathematics. They then react to mathematics through fear or in rigid, rote ways. Their reactions are reinforced by the cultural view that mathematics can only be understood by a select few. (Henderson 1981, p. 12)

Henderson understands how a rote conception of mathematics might develop and might be reinforced. He also understands how this conception of mathematics is linked to the fear and anxiety that many feel in the face of mathematics. Henderson continues,

> Recently, I was thinking back over the times that my perception of mathematics had been changed by the insights or questioning of a person in my class. Suddenly, I realized that in almost all of these cases the other person was a woman or from a different culture from by own. I don't think that this is just a coincidence. (Henderson 1981, p. 13)

Focusing only on the public image of mathematics excludes many voices from the mathematics dialogue and limits our own understanding of our discipline.

A WORD OF CAUTION

Because of their rote conceptions of mathematics, our students often do not hear what we want them to hear. They often DO hear what we leave unsaid and we do not intend for them to hear. We need to reflect on how we view mathematics and be clear about it for ourselves, because we will convey our views to our students, even when we do not intend to. Do we believe that mathematics must be learned by rote? Do we believe that most students are not capable of doing the mathematics we are teaching them? If we believe it, our students will know it. Do we believe that mathematics is a wall separating people by gender or by cultural heritage or by socio-economic background? If so, our students will know it. Are we insecure about the mathematics that we teach? If so, our students will know it. Are we afraid of the creative ideas, insights, or real questions that our students have or might have if we give them the opportunity? If we are, they will know it. We need to be aware of our beliefs. Our students will hear our unspoken beliefs. They are very perceptive.

I have been reading a good bit about death and dying lately. My close friend and professional colleague, Janet, is dying of cancer. Elisabeth Kubler-Ross talks about helping health care professional learn to listen to their dying patients in *To Live until We Say Goodbye*. I found the following passage kept bringing my thoughts back to my students, our students, and to the things our students hear, in spite of what we say, or leave unsaid. Dr. Kubler-Ross writes:

> The purpose of my seminars was to teach young students in the helping professions to take a good hard look at their own fears, their own unfinished business, their own repressed

pains which they often unwillingly projected onto their patients.

Those physicians who were most afraid of the issue of death and dying never revealed the truth to their patients, rationalizing that the patients were not willing to talk about it. These professionals were not able to see the projection of their own fears, their hidden anxiety, yet the patients were able to pick up these feelings and, therefore, never shared their own knowledge with their physicians. This situation left many dying patients in a vacuum, unattended and lonely.

It is also true in our mathematics classes that what we leave unsaid is conveyed to our students. If we believe, but would never say, that mathematics is really hard, or elitist, or something to feel anxious about, they will hear that mathematics is really hard, or elitist, or something to feel anxious about. Many will believe they cannot do the mathematics we are teaching them. Many will believe that they are not the right gender or from the right cultural background to do mathematics. We may ask our students for their questions, but we may have trouble taking their questions seriously, as David Henderson did. They will hear that we do not want to hear their questions or that their questions are not appropriate, or that they must be really dumb to have asked their questions. They will stop asking their questions, but they may be left intimidated, fearful, and wanting to avoid mathematics at all cost. They, too, may be left "in a vacuum, unattended and lonely."

> **"** ***We give them lots of exercises with no words. They hear, "Mathematics is not a language of communication, only computation."***

OUR OWN CONCEPTIONS OF MATHEMATICS

I hope that you are thinking about ways you can listen to your students and help them see both the private and public worlds of mathematics. Let me suggest that you begin by thinking about how you see mathematics. For you, is mathematics most like a maze, a puzzle, a set of tools, or a quilt of intricate design and artistic delight? What is your metaphor for mathematics? How do you fill out this image once you have chosen it? Thinking about your own metaphor for mathematics may help you think about your own conception of our discipline. Then you can continue devising ways to help your students gain a complimentary conception and thinking about the pedagogical and curricular strategies you will use to help them achieve this new conception.

I have for many years asked my students to write their metaphors for mathematics. I have shared a few with you today. [Note: A protocol to gather metaphors in a classroom setting appears in Gibson 1994, pages 11 and 12.] I was delighted to hear what my students thought about mathematics and how frank and honest they were in the metaphors they so freely shared with me. But before their metaphors could be really helpful to me I needed to clarify my own conception of mathematics. I was finally able to express my conception in the form of a metaphor as I ask my students to do. Let me conclude this paper by sharing with you my metaphor for mathematics. I wrote it following my visit to the ACEER Laboratory in the Peruvian Amazon rainforest and my walk on the canopy walkway that allowed me to experience all levels of the rainforest and to climb to the top of the canopy for a perspective from above.

> For me mathematics is like the Amazon rainforest, vast and filled with much that I know and much more that is new or unknown to me. The plants and animals of the rainforest have developed intricate interdependent relationships and adaptations for their mutual survival. By using the canopy walkway and talking with those I meet who point out new things to me and answer my questions, I come to appreciate the variety and differences at all levels of the rainforest and finally stand above the canopy to gain perspective. I read, observe and question; I experience and I touch; I share my knowledge with others always listening for their perspective and understanding. While I have much to learn about the plants, birds, and insects, and their interdependence on each other, I can appreciate the whole that they have collectively created. The rainforest is vast, sometimes thick and dark, other times

> quite open; it is empowering, not frightening. On each visit I see more, understand more, and feel more connected with nature, knowledge, and myself. (Buerk 1996, p. 27)

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Math Induction

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Adapted from Camp, Dane R. "Math Induction" in Math Song Sing-a-long, edited by John A. Carter and Dane R. Camp. Booklet presented at Illinois Council of Teachers of Mathematics Annual Conference, Springfield, Illinois, 1998. May be sung to the tune of "Blowing in the Wind" by Bob Dylan.

How can you prove that a statement is true
For any counting number n?
Cause there's no way you could try them all—
Why you could barely begin!
Is there a tool that can free us
From this quand'ry we're in?
The answer, my friend, is math induction,
The answer is math induction!

First you must find an initial case
For which the statement is true
Then you must show that if it's true for K,
Then K+1 must work, too!
then all statements fall like dominoes
Tell me, how did we score this coup?
The answer, my friend, is math induction,
The answer is math induction!

The Word Problem and the Child

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After seven consecutive years of working with about 150 children (per year) solving word problems, I feel some understanding for what is happening...[This article] represents a college professor's view of children doing math.

INTRODUCTION

This article is concerned with a national problem in math education, namely, children after the third grade seem to show little interest in learning math by drill and practice. As the children grow up, they seem to remember very little of the math which they have seen. When they become college age, either they are in need of remedial math for college or they bring a weak math side to the large service sector of our society. We maintain that it will make all the difference in the math history of each child if a weekly junction can be made between the side of the child that does elementary word problems and the side of the child that handles any challenging intellectual activity. Exposing children to algorithm after algorithm or calculator solutions does not really cultivate the problem solving (or math) state of mind. What is more attractive and, hence, memorable, is the pattern which is being exploited by the algorithm or the calculator. Patterns and accompanying ideas are far more likely to capture and hold the interest of children. In a sense, patterns need to be discovered before they are conceptualized and committed to memory. Such discoveries come naturally when the child goes on a math pilgrimage which leads to the solution of a word problem.

There is no solid reason why children should not try to solve word problems every week for most of the year. Elementary problems can be solved with very few steps of math reasoning. Such word problems use the syntax of ordinary language, not some special syntax of math (or science). Also, any child has a personal history filled with trying to cultivate language. MIT professor, Steven Pinker, in his best selling book, "The Language Instinct," documents the existence of a strong instinct for language acquisition in every child until age six. Pinker's mentor, Noam Chomsky, said "Children develop these complex grammars rapidly and without formal instruction and grow up to give consistent interpretations to novel sentence constructions that they have never before encountered." Hence, a persistent and purposeful teacher who takes the time and energy to communicate word problems in a free flowing language which is a mixture of both math and English has a great ally inside the child, namely, an activated language instinct.

A WORD PROBLEM PER SE

As a point of reference for the rest of this article, we now offer an example of what we mean by an elementary word problem.

A Boston policeman has a work schedule which consists of working four consecutive days, then having two days off. Assume this pattern continues throughout the year, 2000, which is a leap year.

What is the exact number of days such a policeman will work during 2000?

Problems such as this, when solved using only your wits and imagination over an extended period of time, offer the best way for children to learn and to remember the math of the counting numbers: 1, 2, 3, 4, and so on. Words such as policeman, work, and schedule engage the child's conscious, if not the subconscious, helping the child get close to the problem. The presentation of the problem is common enough. There is some news (as in the newspaper or on the internet), stated in the third person, but some information seems omitted, and there is a question to answer. Initially, the child wants to guess at the answer, which is a nice way to break any paralysis. There then comes an attempt to penetrate the problem by seeing what physical world phenomena are being measured by counting numbers. A veteran problem solver tries to see if the words of the problem speak to some math counterpart inside himself or herself. Such math counterparts are built up over time. For example, patterns of numbers such as the multiples of six are constantly associated with a specific choice of words, say read aloud as 6, 12, 18, and so on, not unlike the words to a song. The pattern-word combination eventually becomes part of the child's recent experience with math. There is a noticeable look in the child's eye when a math pattern from the recent past is recognized. Once it is observed that multiples of six are involved, say in the given problem, the child needs the courage and self-confidence to try to do some math in spite of insecurities. Eventually, the child will reach a point where the insights that 366 is 61 times 6 and 61 times 4 is 244 days give the answer.

It does not take long for the child to realize that although the statement of an elementary word problem has a common appearance and the problem typically has one correct answer, there are many ways to approach the solution based on each individual's personal history. One thing which does not vary is the idea that numbers and patterns of numbers are "anchors" that one goes to when searching for a solution. Another regular feature is the reward that comes to the successful problem solver, i.e., a tremendous surge in energy which is reminiscent of the expression "innate enthusiasm."

BYPRODUCTS OF REGULARLY SOLVING WORD PROBLEMS

Although some children will take longer than others in acquiring the knack of solving such problems, each child is capable of acquiring the knack of solving such problems, given an appropriate setting. Also, it is worth the effort. This is because this activity puts you in touch with your creative side, makes you fashion conjectures and build frames of reference. It also gets you to gather yourself together, to move over to where the problem is as opposed to reaching from a distance, to attach yourself to the problem, making the problem into a personal companion, to exert self-control by choosing certain options over others, and, in general, to learn to leverage the integration powers of your brain to get the most production. This means that the child will visit often and become familiar with a certain state of mind. This state of mind becomes an intellectual force which can be brought to bear on non-math problems as well, such as how to hit a baseball, or, later in life, how to get a very first job. In fact, for some children whose brain is especially suited to it, this state of mind can even develop into a kind of philosophy of life where one sees any obstacle as a problem to solve.

Something remarkable happens when this problem-solving state of mind is brought to bear on elementary word problems. The child gradually appreciates how to penetrate a context to fashion a bulls-eye, and to focus on the bulls-eye when both the context and the bulls-eye involve numbers. The child's natural imagination fuels efforts to attain proficiency in trial and error with numbers, to adjust guesses based on computations, to avoid paralysis while stumbling through reasoning about the natural numbers, and to try to find the right collection of numbers with which to deal. The energy to do this comes from the child's creative side; the resulting satisfaction, if not exhilaration, disposes the child to try to have the same experience again. In particular, the child can get a big kick out of connecting new patterns of numbers with old patterns and seeing how they fit together. All this results in a truly amazing evolution of a rookie problem solver into a veteran problem solver. Cultivating the brain's integrating ability to help solve elementary word problems actually develops a basic operating structure for doing new math in the future. Finally, the child will obtain a new view of the physical world and this view will be confirmed by getting additional problems correct.

THE MILIEU FOR CHILDREN SOLVING WORD PROBLEMS

Having indicated how solving word problems can help children be more self-sufficient in their later math efforts, we next address the issue of resistance to internalizing math information as it is typically presented today. What is offered is based on seven consecutive years of experience with a program being implemented at six different elementary schools in Boston, MA. It amounts to trying to maintain the children's interest through a social setting which gives due respect to both the nature of children and the nature of mathematics.

The format is a combination of a math team, practice once a week, and a contest once a month for five months (provided by George Lenchner's invention: Math Olympiads for Elementary Schools). Each practice lasts forty minutes and consists of going through seven word problems at the rate of five or six minutes per problem. Consecutive word problems may be independent of each other, so math information acquired in any one session will be modest and will be seen in a disconnected fashion. The practices feature lots of questions and answers, no one of which solves a problem but together point the way to a solution of the problem. A major role is played by the reality that hearsay in math is more reliable than hearsay in day to day life. Second and third hand information can be of good quality even though the messenger did not discover the information for himself or herself. The children learn directly from the problems and from each other; they learn indirectly from the teacher through the choice of problems and the little things in math that are emphasized. Concerning the monthly contests which had five problems to be done with no help from anyone, children who were rookie third graders were told to pick out one of the problems that appealed to them and to spend time on that one problem. Children who were veterans were told to start with whatever problem appealed to them, then go on from there.

As far as accommodations made giving due respect to the nature of children, the idea is that when children strain themselves creatively and get lost in a problem, they lose their balance. Thus, they need a gentle structure that will easily allow them to regain their balance. The environment should be one of utter civility, one void of extreme or dramatic action except when channeled into a math effort. All conversation should carry an air of dignity and respect, even those statements which are full of substance should be delivered with moderation.

It is in the nature of a child to talk about and compare problems with another child, perhaps even tease each other that one got the correct answer and the other did not. This conjures up the image of an environment which allows for a genuine sense of camaraderie among the children. Ideally, one would look for contributions to be made by those children who are presently not yet as skilled in math. This kind of child can add some inspiration to the group or try to encourage a sense of team spirit while working on a problem. For another example, a child's attention span is anything but constant; it rises, falls, grows, declines, and expands inconsistently. When working on problems, the environment should cultivate "volunteerism." A child not disposed to assert himself or herself should readily see examples of other children trying to assert themselves; a child should see examples of children with self-confidence and then aspire to have self-confidence too. As a final example, children tend to live in a world of impressions and ideas; formal technical behavior is foreign to them. Hence, the environment should value good ideas and partial solutions which are children's inventions even if they are not directly relevant.

CONCLUSION

In order to put this article in perspective, let us relate its content to well known large scale efforts in math education. The NCTM Standards of 1989 are being revised, and the draft of the new NCTM Standards is now available on the web. In Massachusetts, there is the Mathematics Curriculum Frameworks which was written in reference to the NCTM Standards. The core concept of the Frameworks is "Advancing Mathematical Power by having the students cultivate problem solving, communication, reasoning, and connections." One view of this article is that it attends to the Frameworks in a narrow (or compact), but organized way as far as certain problem solving is concerned. Another view is that it tries to get at the "why" of the Frameworks from the author's own experience of observing children doing elementary word problems for seven consecutive years.

In other words, thoughtful documents such as NCTM Standards or Massachusetts Frameworks still need to be interpreted for elementary school principals. The principals need something like a Global Positioning Receiver to help navigate the landscape of doing word problems. This article offers some help to such principals.

On another level, children are unfinished in many ways, but are growing and maturing, even quite apart from their teachers. Their ability to reason is naïve, perhaps, analogous to the way the immune system of very young children is naïve. Such children need help in correcting subjective and careless reasoning; they have to learn how to sit down by themselves and figure something out. In a broader setting, children need to understand that learning is valuable and to see that "real" learning and "school" learning are the same. This article offers a device that has done this for children in the past.

Finally, there are questions that are ignored in this article. One such question is how intrusively should technology be inserted into the problem solving context. A deeper question is what role does the subconscious play in learning math through word problems, or, in general, in any challenging intellectual activity. Partial answers to these questions may very well come from similar endeavors as this one i.e., a math educator attending to certain aspects of the NCTM Standards or the Massachusetts Frameworks.

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Written to Me Upon Getting a B in Linear Algebra

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A revised version of a poem by Serge J. Zaroodny

Like a warrior returning from a fight,
  Who doth return himself, but minus foot,
The which his victory redeemeth not aright,
  For he would rather have that, than his loot.

Or like a knight returning from a fray
  Wherein his mighty foe he did o'erwhelm,
Yet cannot bid his thoughts to cease to stray
  To broken shield, dead horse, and ruptur'd helm;

Or like a boxer coming from the ring,
  Where the remains of his opponent lie,
His champ'onship deems well worth suffering,
  Yet feels his head and ribs and hopes to die;

Ah, sweet the quest, but better yet without it!
My Linear Algebra! Thus do I feel about it!

The Need for Interviews in the Mathematics Classroom

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The following task was given as a warm-up activity by one of my student teachers (Lenny) to a group of eighth-graders:

A baker used two-thirds of the flour that he had to make a cake, and two-thirds of the remainder to make bread. If he then had two-thirds of a pound left, how many pounds of flour did he have at first?

The students were required to obtain a solution and be prepared to explain their solutions.

Two of the many solutions were as follows:

1. $\frac{2}{3} + \frac{2}{3} + \frac{2}{3} = 2 \text{x} 3 = 6$

2. $\frac{2}{3} \text{x} \frac{2}{3} = 2 \text{x} 3 = 3 \text{x} 2 = 6$

“*To understand the thinking of children, teachers need to spend more time listening to children describe how they think and less time explaining to the children how the teacher thinks.*

Both results are correct (ignoring the unit in the answers). These solutions bring certain questions to mind. How many points out of five will you give for each? Would you follow-up these solutions with the students? How would you follow-up?

It may be a good idea to follow-up such situations with a 1-1 discussion with the student requiring him/her to explain his/her solution. While the work seems weird, the answers are correct. It is possible that the student might have used a trial-and-error approach and obtained the correct answer; the student then tried to justify the answer by showing some work (because the teacher requested it). How do we know what occurred? According to Rudnitsky et al. (1981), teachers' understanding of what a child knows is derived from dialogue with the child. Unless we talk with the students, asking Why? How? and What?, it is difficult to determine what thinking went into the solutions. "To understand the thinking of children, teachers need to spend more time listening to children describe how they think and less time explaining to the children how the teacher thinks" (Chambers, 1995, p. 380).

The claim in this paper is that by talking with the student, we probe into his/her mind to understand the thought processes; thus we are able to identify the student's specific difficulties and place ourselves in a better position to help the student. According to Huinker (1993), "Interviews are a method of assessment that allow us to gain insight into students' conceptual knowledge and reasoning during problem solving. With paper-and-pencil tasks, students' understanding is often hidden" (p. 80). The student benefits from the experience by being able to clarify and communicate his/her thoughts. Buschman (1995), for example, states that "When students write or talk about mathematics problems, they test, expand, and extend their understanding of mathematics" (p. 329). The National Council of Teachers of Mathematics [NCTM] (1989) makes similar claims by saying that "Communicating helps children to clarify their thinking and sharpen their understandings . . . [P]robing questions that encourage children to think and explain their thinking orally or in writing help them to understand more clearly the ideas they are expressing" (pp. 26-27).

Situations in which students give the correct answer for the wrong reason are not unknown. There is the well-known example

$$\frac{2\not{6}}{\not{6}5} = \frac{2}{5}.$$

[For more examples and a discussion on this see Thomas (1967); Carman (1971); Shaw & Pelosi (1983); Borasi (1986).] It is important for students to give the correct reasons for their answers. To ensure this we have to require them to explain their solutions. During our lessons we should ask them How? Why? and What? questions. These questions do not necessarily have to follow students' incorrect answers; it could be equally informative to follow-up correct answers with questions. Hollander (1977) feels that opportunities should be available to discuss the rationale for correct solutions.

Suppose a student is asked to reduce the expression

$$\frac{3a + 15}{4a + 20}$$

to its lowest terms and the student gives

$$\frac{3a + 15}{4a + 20} = \frac{3}{4}$$

as the answer; the teacher may not require an explanation from the student because (3/4) is the correct answer. However, it could be useful to ask the student to explain how (3/4) was obtained because it could have been obtained by incorrect work. ['Cancel' the *a*s, and 'cancel' 5 in 15 and 20; that leaves

$$\frac{3+3}{4+4} = \frac{6}{8} = \frac{3}{4}.]$$

Shaw and Pelosi (1983) discussed an interesting example involving arithmetical division. These examples also point to the inadequacy of written work to explain students' thought processes.

The practice among mathematicians to ask themselves and their students How? Why? and What? questions is not new. It dates to the time of the ancient Greeks who, as a result of asking these questions, developed deductive arguments. More recently, support for interviews, dialogue with students, and for requiring students to explain their work have come from Weaver (1955); Lankford (1974); Pincus et al. (1975); McAloon (1979); Schoen (1979); Rudnitsky et al. (1981); Brownell (1987); Liedtke (1988); Lampert (1988); Whitin (1989); NCTM (1989, 1991); and Huinker (1993). For example, according to NCTM (1991), "Paper-and-pencil tests, although one useful medium for judging some aspects of students' mathematical knowledge, cannot suffice to provide teachers with the insights they need about their students' understandings in order to make instruction as effectively responsive as possible . . . [I]nterviews with individual students will . . . provide information about students' conceptual and procedural understanding" (pp. 63-64). (For more supporting references see the February 1995 issue of *Teaching Children Mathematics*.)

Most of these writers have suggested that interviews can be used as a diagnostic technique. However, it can be used also as part of the teaching process to obtain feedback on students' progress. For example, during a lesson the teacher can ask students to explain how they arrived at their answers to questions. Based on their responses, the teacher can decide how to proceed with the lesson or what course of action to take. Diagnosis can be followed by the preparation of remediation and/or differentiated programs of instruction which would enable students to overcome their difficulties. In the United Kingdom, Booth (1984) studied, in great depth, through interviews of students, some of the errors in mathematics which had been identified by Hart (1981) and then designed teaching experiments to correct these errors. The experiments were successful.

The following are some of the purposes/advantages of interviews in the classroom:

(a) to identify students' difficulties and to ascertain the reason for the difficulties;
(b) to probe into the learners' thought processes to find out how they are thinking and reasoning;
(c) to obtain feedback on students' progress;
(d) to provide opportunities for students to communicate mathematics and for them to clarify their thinking about mathematical issues;
(e) to help students identify and correct their mistakes;
(f) to provide opportunities for students to justify/defend their arguments;
(g) to determine whether the learner has a correct reason for his/her answer;
(h) to find out what students know and understand;
(i) to obtain information which would direct the planning of remediation/differentiated programs.

The use of interviews in the classroom should not be misconstrued as being problem-free. Interviews may yield inaccurate information. The interviewee may give information that he/she thinks the interviewer is looking for and may also fail to recall information accurately. An interview is obviously a time-consuming activity and may require trained personnel to conduct it. When students are asked to explain their answers and solutions during a lesson, much time is utilized. The teacher runs the risk of not completing the lesson or program of work. The good things are that the teacher does not have to question every student, and some of this questioning can be done outside of class time. Also, a student may be interviewed while the others are working. Technology could be helpful in diagnosing students' difficulties (Ronau, 1986). It seems that the long-term benefits of interviews will outweigh the initial disadvantages. Huinker (1993) recognizes some of the difficulties associated with interviews but feels that these difficulties can be overcome with "careful planning and organization" (p. 81). She identifies some important points to consider before, during, and after an interview and provides useful ideas for conducting interviews.

Using interviews in the classroom is not an entirely new idea. This practice has been used in the past with some success but, for one reason or another, interest in it waned. Students' written work alone is inadequate to determine students' thinking. Dialogue with students is potentially efficient in diagnosing students' specific difficulties. Once these difficulties have been identified, appropriate programs of instruction can be planned for students. The objective of teaching is to optimize learning, and one way to achieve this objective is to understand the thinking of students. The time has come for us to renew our interest in the practice of using interviews in the mathematics classroom in order to make our teaching more effective and to encourage communication of mathematics, one of NCTM's Standards.

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A Glorious Constant

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"In a riddle whose answer is chess, what is the only prohibited word?" I thought a moment and replied, "The word chess." —Jorge Luis Borges, "The Garden of Forking Paths"

Without naming my burning passion,
I shall sing a song of admiration,
A glorious hymn to a constant,
A basis for natural logarithms
That most of us know from school.
***
—But why?—you ask sarcastically—
Such a crazy soliloquy
To an ugly irrational quantity
Which could not possibly function
As a root of a good-looking polynomial?
So I shall proclaim loudly
Its validity in day-by-day affairs
Along with its bright shining
Throughout human history.
In truth (in all probability),
Pythagoras was ignorant of it,
And it was also unknown
To that miraculous Syracusan,
To philosophizing scholars of Islam,
To giants of Indian astronomy,
To pupils of famous Confucius
In kingdoms of classical China.
This brainchild of a prolific Swiss author
Was first shown to scholarly public
In *Introductio in Analysin Infinitorum*,
Occurring again and again
In his many works in analysis
And also in various manuscripts
Of his illustrious rivals.
***
Today, that fantastic constant
Is vigorous, strong, and mighty,
Occupying its rightful position
In our faithful minds;
Not only in abstract thinking
Or among old and dusty
Gloomy and dry formulas
In old calculus books;
It puts up a lot of labor
Within today's banking,
Supporting that branch of industry,
Its living spirit and soul.
For, anybody in favor
Of continuous compounding
Of his growing holdings
In a fat savings account
Will always call for that constant,
Writing its charming symbol
(Fifth from a starting location
On a traditional Roman list).
***
—But how big is your constant?
With an aim of approximation
(A smart and studious school kid
Could accomplish that task so fast),
Put various factorials
In bottom parts of fractions
In which our familiar unity
Is invariably on top,
Sum up all such fractions,
Going in that fashion
As far as you may wish,
This way you obtain quickly
A fairly good accuracy
In approaching your diamond goal.
***
Now, would you discuss for an instant
A truly profound topic
(Which many find mildly amusing),
A highly dramatic fact:
Raising our gracious constant
To what amounts to a product
Of an imaginary unit
And a circular constant π,
It would stand in stark opposition,
To a multiplication unity,
Displaying a minus sign.
And that our Swiss advisor,
With an utmost clarity,
Saw in his brilliant opus
During his Prussian sojourn.
***
To this all-inspiring constant
I sing a song of glory,
That nobody could rival
In annals of misty past;
It knows no spatial limits,
Will last for thousands of autumns,
And nobody could inhibit
Its triumph in any world.

Using Environmental News to Help Teach Mathematics

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Do you teach at a school where there is an opportunity to develop new courses in environmental mathematics? Is there time available in your courses to develop major projects involving building models of environmental problems? Those who can answer "yes" to such questions are fortunate indeed. But even if your answers are "no," it is possible for those of us with an interest in the environment to use this topic in a wide variety of courses in such a way as to help teach math, develop quantitative literacy, and legitimately put environmental considerations into the curriculum.

One approach is to keep a file a clippings of items from newspapers, magazines, journals and other information sources that relate some aspect of the environment to some aspect of mathematics. There are a huge number of possibilities, as will be illustrated. Gradually get used to adapting such items for classroom use; eventually you will make frequent use of such items. They might lead to a simple homework problem or exam question, or illustrate some point in a lecture, or help generate classroom discussions. You do not have to wait until you have the time for a big project or a new course.

I believe you cannot depend on the textbooks for this. Though it is the fashion nowadays for textbooks to use real data and emphasize applications in many lower division college math courses, and rightfully so, texts doing this may try to cover a wide variety of applications, so cannot often include the environment. Even when they do, it may be that the examples may not do much to encourage interest in either the environmental topic, or for that matter, in the mathematical topic.

For example, if you look in the "index of applications" in one mainstream text we use you will find a reference to "Recycling," which turns out to be this problem:

Let U = set of all participants in a consumer behavior survey conducted by a national polling group.
A = set of consumers who avoid buying a product because it is not recyclable
B = consumers who use cloth rather than disposable diapers
C = consumers who boycotted a company's products because of their record on the environment
D = consumers who voluntarily recycled their garbage

The problem requests a verbal description of *A* intersection *C*, *A* union *D*, *B* complement intersection *D* etc. The author could at least have used set *C* for the Cloth users and *B* for the Boycotters to help keep the categories straight! And while it is nice to raise these recycling issues, who really cares about these unions and intersections? Though students do need drill, I do not see how this problem will generate interest in either set theory or recycling.

Another example from a competing text that has one reference to "smog control" in its "index of applications":

A new smog control device will reduce the output of sulfur oxides from automobile exhaust. It is estimated that the rate of savings to the community from the use of this device will be approximated by $S(x) = -x^2+4x+8$ *where* $S(x)$ *is the savings (in millions of dollars) after x years of use of the device. The new device cuts down on the production of sulfur oxides but it causes an increase in the production of nitrous oxides. The rate of additional cost (in millions) to the community after x years is approximated by* $C(x) = 3x^2/25$.

The question goes on to ask how many years one should use this device and how much can be saved. It is good to give students practice with such problems, and help make them aware of costs, benefits (especially that there might be dollar benefits from pollution removal!) and trade-offs, but it is evidently a pretty artificial problem. Who estimated such a formula? We know how sensitive that is to the interests of the estimators! For what interval of values of x might this be good? Not stated. Those who follow pollution issues know that cars are not a major source of sulfur dioxide. The prediction of $12 million in benefits after 2 years but a loss of $52 million after 10 years (within the lifetime of an automobile) sounds implausible.

Another problem from the second book, indexed under "pollution:"

Pollution from a factory is entering a lake. The rate of concentration of the pollutant at time t is given by $P(t) = 140t^{5/2}$ *where t is the number of years since the factory started introducing pollutants into the lake. Ecologists estimate that the lake can accept a total level of pollution of 4850 units before all the fish life in the lake ends. Can the factory operate for 4 yr without killing all the fish in the lake?*

Does this problem suggest it is OK to kill 90% of the fish in the lake? The pollutant is unnamed, and the author does not even bother identifying the units of the pollutant. I doubt this can help develop much interest in either the environment or in mathematics.

Contrast that example with this one from the Fall 1993 newsletter *Science for Democratic Action* which has a regular section called "Arithmetic for Activists."

You live one mile downwind of a uranium mill. Your trusty air monitoring equipment measured the amount of radioactivity in the air. You read 0.00037 becquerels per liter of air. Remember that 1 curie = 37 billion becquerels and that the prefix 'pico' means one trillionth. Laboratory analysis indicates this is all due to insoluble radium-228—are you above the standard?

It is noted that the existing standard of insoluble radium-228 is .001 picocuries per liter.

This time there is no escaping the need to pay attention to units. [Though we do not mine for uranium in New England, I do like this problem. Our region is relatively dependent on nuclear energy but almost nobody pays any attention to the details.] This problem is copied to my file, so it can be used in a "technical" math or quantitative literacy class.

I think if we are to get good real examples, it is clear we cannot depend on textbooks but should expect to gather them ourselves from a variety of sources.

Though even beginning liberal arts statistics courses try to cover as much statistical inference as possible, we are more likely to be training citizens in these courses than researchers. Therefore, on the first day of the course I mention the most basic interpretation of the word "statistics" as a set of meaningful facts and figures. To illustrate this, I show a slide of one of a "Harper's Index," a monthly compilation of interesting statistics in that sense. The students see that numbers can indeed be interesting, not only in the judgement of statisticians, but in the judgement of magazine editors who are in the business of selling magazines. Indeed the "Harper's Index" is successful enough to be a registered trademark. There is even a "Harper's Index Book" paperback! Examples from the February 1997 Harper's Index include:

Amount that "side agreements" in NAFTA require that the U.S. spend on environmental cleanup: $1,500,000,000
Amount the US had spent by the end of 1996: 0

Older indices include:
The number of Exxon Valdez spills it would take to equal the amount of oil spilled into the Mediterranean each year: 17;
Estimated percentage of the $6.7 billion spent on Superfund cleanups since 1980 that has gone to lawyers: 85;
Square yards of park per inhabitant: Paris 6, New York 18.

You get the idea—some amusement, but also some seriousness. If you flash a slide of an Index on a screen and ask students what is interesting about the items, they may well single out those with an environmental theme. The Indices also include numerous percentages, averages, probabilities, and ranks, but the idea is not so much to teach mathematical terminology as to encourage students to develop a lifelong belief that quantitative information is interesting and worth paying attention to.

One actually needs to clip the "Harper's Index" pages to have it available when needed. One of my colleagues who knows I use this sort of thing actually gives me a copy of the "Index" each month, and I occasionally get similar materials from other colleagues and students who know I like to collect such information. Many others use the idea of the "Harper's Index." For example I have a "Vital Statistics" page from the National Wildlife Federation which includes these items:

Estimated global pesticide sales in 1975: $5 billion, in 1990: $50 billion
Parts per million of DDT in human adipose tissue in the US in 1970: 8 in 1983: 2

See? Do not always assume bad news!

"Harper's Indices" have numerous references to very large numbers. This is one of the first topics developed by John Allen Paulos in his book *Innumeracy*, (indeed it is referenced in the second line of the book!) because of the difficulty even educated people have in dealing with numbers in the billions, trillions etc. Think of the problem mentioned before about the picocuries. Do you have colleagues who tend to refer vaguely to zillions? It seems even newspaper headline writers do not pay adequate attention. For example, I clipped a headline from the May 16, 1984 Providence Journal that says "Waste Cleanup Cost: up to $26 million." We wish it was $26 million! More than that was spent on just one superfund site in Rhode Island, the Piccillo Pig Farm. The article clearly says the cost was up to $26 billion, but apparently millions and billions were all the same to the headline writer.

What we can we do to humanize the $26 billion figure? As it was supposed to be spent over a twenty year period to clean up the sites, one could ask what it would cost on average per person, per year. One thing I like about that question is that you have to divide twice. Textbook problems illustrating the mean never seem to have such questions even though there are numerous real situations where it applies, and some students are puzzled about what to do. Another thing I like about the question is that the answer comes out so small, only about $5 per person per year. Indeed one student told me he thought it must be wrong, it was so cheap. Perhaps that is how we should argue before Congress when debating spending money on hazardous waste cleanups.

Humanizing large numbers by reducing them to a per person or per household basis as done above can be applied to a variety of situations. The process can also be reversed to see the cumulative impact of what sounds like insignificantly small numbers. For example, because of my interest in the impact of transportation on the environment, I get a lot of information on that topic from a variety of sources. Parking cashout is a strategy to reduce vehicle miles by having employers who offer free parking also offer the cash value of the parking as an alternative for employees who don't use it. A parking cashout leaflet from the Conservation Law Foundation (based on a study at the UCLA School of Public Policy and Research) suggested that parking cashout can reduce auto commuting by about 625 vehicle miles per month per employee. Assuming each mile of auto use produces about .4 ounces of carbon monoxide and .038 ounces of nitrogen oxide emissions, which sound insignificant, one can ask for a reasonable estimate of the effect of how a national parking cashout program might influence the total weight of the output of these pollutants. One would need an estimate of the total labor force, and what percentage gets free parking. A ballpark estimate for the total might be of order of magnitude of a billion pounds of carbon monoxide and about 100 million pounds of the nitrogen oxides, which does sound significant.

Another example of going from a small human scale number to a large number relates to solid waste. This is a topic that has gotten much attention here due to problems at the state's central landfill (literally one of the high points of Rhode Island. Look for it if you ever fly in to Providence) and several attempts to build solid waste incinerators, which alarmed people living near the proposed sites. I relate this to a problem in one of our current textbooks (in our math for elementary teachers course) asking for the surface area of a cereal box of dimensions 11" by 2.5" by 8". A check of my favorite cereal box shows that these dimensions are realistic. But instead of stopping at the answer to the textbook question (271 sq in), why not go on to consider the surface area of a cube that would enclose the same volume? It may surprise some students that the same volume can be enclosed by only about 219 sq inches of box, a savings of 52 sq in or about 19.2%. Ask for a reasonable estimate of the number of cereal boxes sold in a year (100 million households times 50 boxes a year per household was suggested) to get an estimate of the total reduction in packaging possible from redesigning cereal boxes. Ask the class why isn't cereal sold in the shape of a cube. Would they buy a box of cereal in the shape of a cube? What happens if the dimensions are modified to make it only somewhat more cubelike?

There are marketing considerations here; the cereal companies want to have a large face area on which to show their brand name and logo. I do think, and sometimes say, that the cereal producers do not have to worry much about the cost of disposing of the empty boxes. At any rate, it is helpful to have a file of solid waste clippings available. For example, one such article, perhaps relevant to the cereal box example, indicates paper and paperboard constitute 40% of our waste stream, a considerable percentage. It also says that the Northeastern Governors have asked industry for voluntary cooperation in reducing packaging though none of my students thought that industry would actually pay any attention to such a request.

The cereal box example makes a good multistep problem, starting just with the dimensions of the cereal box. Every math teacher knows students will do OK on one step problems—they learn the procedure—but do badly on multistep problems even if they really know how to do each step. There just isn't enough practice with such problems. We math teachers have to be on the lookout for them ourselves.

That example also opens up the possibility of a percentage of percentage question. What would be the impact of a 19% reduction in 50% of the 40% of our solid waste stream? In the ideal world all students would be able to deal with percentage problems, but in the world I live in that topic is apparently not well reinforced in the usual high school math curriculum of Algebra I, II, and Geometry. So many students need help, and practice, on percentage problems. Indeed the September 1995 Harper's Index notes the chances that an American seventeen year old can express 9/100 as a percentage is 1 in 2. So solid waste issues can lead to many percentage problems. An example I've used relates to the fact that Rhode Island, unlike our neighboring states, does not have a "bottle bill"—that is, mandated deposits on beverage containers. When that was being debated here a flyer used by the Bottle Bill Coalition included this item that I used for this percentage problem: "According to *Beverage Industry Magazine,* the US soft drink companies spend $4.5 billion annually for packaging their product and about $800 million for ingredients. What percentage of the total of these costs is for packaging?" Sadly, many students didn't handle this correctly, dealing badly with the percentage or with the large numbers involved. Also sadly, the item didn't help pass the bottle bill, which our neighboring states have found to reduce litter and increase the recycling rate.

"
Indeed the September 1995 Harper's Index notes the chances that an American seventeen year old can express 9/100 as a percentage is 1 in 2.

I have in my solid waste file a full page ad from the American Plastics Council entitled "Plastics. An Important Part of your Healthy Diet." I haven't decided how to actually use it, if at all. What is significant is that it has no quantitative information whatsoever.

While on the subject of relating human scale activities to environmental issues, consider this example. Amtrak, our intercity rail passenger service, is facing severe financial constraints and has asked that one half cent of the gas tax be set aside for capital improvements for Amtrak. There is a letter to the editor from some motorist objecting to how their hard earned dollars would be going to a service they do not use. I suggest asking for a quick guess of how much this will actually cost a typical motorist. Then ask for a procedure to come up with a more reliable estimate. For example, if you drive 15,000 miles per year in a vehicle that gets 20 mpg, the total cost of the half cent tax would be about $3.75/year which a student said was "nothing." Indeed it does seem like quite a bargain to keep a rail passenger system alive that might someday provide an alternative for even the most train-phobic motorists, or at least help reduce traffic on the roads for those who must drive. But the point to make to students is: "do a calculation to see what it means to you."

By the way, I've found that even environmentalists are often unaware of just how much Amtrak survival is an environmental issue. I have an article indicating the pollution in grams of hydrocarbon emission per passenger mile is about .1 for the rail mode, .2 for buses, and 2.1 for a single occupancy auto. There are similar figures for carbon monoxide. There is also fuel efficiency and all the impacts that energy extraction and transport have on the environment. Amtrak says it can carry 550 passengers one mile on only 2 gallons of fuel while 110 five passenger cars would take 8 gallons to do this. This could turn into various questions for a class—what does this assume about the automobile mileage? If true, what is the macroscale impact?

I do not want anyone to think only information from environmental groups and articles that make some kind of pro-environmental point are to be used. The American Automobile Association puts out annually a summary of the costs for operating an automobile. One of these says the cost of owning and operating a new car (for 15,000 miles) is "now averaging 38.7 cents per mile... Average per-mile cost is determined by combining operating and fixed costs... motorists nationally paid an average of 9.16 cents per mile in operating cost (gasoline, oil, maintenance, tires). Fixed costs, which include insurance, depreciation, registration, taxes and financing, average $12.14/day." I use this in our first "quantitative business methods" course, asking students to read the article, develop the formula, and do some calculations. There is some need to be careful about units—it's amazing that sometimes students do not distinguish correctly between dollars and cents—in asking them to verify the formula or estimate the cost of driving 10,000 miles in a year. A larger environmental point can be made by looking at the percentage of the total cost of driving, that is the marginal or operating cost, which is only about 17%. I believe the relatively high fixed cost but low marginal cost is related to the difficulty of reducing vehicle trips, that is, it doesn't cost very much more to do a little more driving.

An example I use to illustrate functions and marginal costs is taken from information I've filed from the New England Power Company that generates electricity for our local utility. It has a graph of the cost (to the power company) of removing pollution from the stacks as a function of the percentage of pollution removed. Of course it rises quite steeply after a while, actually after about 78% removal. Some students may be disappointed about what that implies for simply requiring 100% removal. Perhaps it is just as well the Congress of the 1970s did not see such graphs when they were writing the clean water laws intending to eliminate all discharges.

Energy issues are a good source of mathematical ideas. First, numerous data lists have been published, with meganumbers sometimes making interesting points. An example from our local paper is that the cost of the nuclear plant in Seabrook, NH, is enough to pay for all municipal services in Fall River for 51 years!

New York's Con Edison utility had an energy quiz in the *New York Times*. A sample question follows. Do you know the answer?

Burning the oil required to light one ordinary 75 watt bulb for a year releases how many pounds of gases that might contribute to environmental problems? (a) 275 pounds (b) 5 pounds (c) 12 pounds (d) 1 pound.

Con Ed said the answer is (a). A less benign example from our gas here is in its rate structure. A handout given to customers describes this monthly commercial rate structure: "Customer Charge is $8.15. First 4000 ccf 63.57 cents per ccf. Over 4000 ccf: 55.76 cents per ccf." This gives us math teachers the opportunity to discuss not only a real piece-wise defined function (writing it algebraically poses more of a challenge for students than I had anticipated) but also marginal costs and the effect of a declining rate structure on the conservation ethic.

Another energy item from the New York Times relates to power line electromagnetic fields and cancer. The headline reads "Federal Panel Says Electric Fields Pose No Known Hazards," but the article itself says that "the statistically weak link between leukemia and proximity of large power lines may be due to unknown factors with no connection to electromagnetic fields. Possible outside factors which need to be looked into more closely include the age of homes and their construction features, pollution, local air quality and heavy traffic near power lines..." This can be a good springboard for a discussion of confounding in statistics. I can also say that, despite the headline, the article did not encourage my students to locate near a power line.

Being on the lookout for examples of confounding in an environmental context can be rewarding. For example, the students see the point instantly in a *New York Times Week in Review* headline "Reading at 55 Miles Per Hour" which reports that accidents were 41% higher when billboards were around. It would be nice to use this to justify removing the billboards and so improving the scenery, but it can be simply that advertisers prefer them where there is heavy traffic.

Population growth models are a staple of environmental mathematics, and there are plenty of graphs from environmental groups showing exponential type growth. A more unusual graph is from the 11/17/96 *New York Times* headed "The Population Explosion Slows Down." The "today" point on the graph is just where the rate of increase starts dropping. It came just at the right time for my Calculus I class studying the second derivative. When I put up a slide of this graph, they immediately saw it as a point of inflection. However, anyone using this beware: the slowdown in growth is only a projection!

Do not think always being on the lookout for examples that can be used in classes will make it too hard to just relax and read the paper. I believe it is worth some effort to be able to use a wide variety of topics. Here are some miscellaneous examples to show this:

Reading that Sarawak (in Malaysia) had 21,000 square miles of rain forest but it was being destroyed at a rate of 1000 square miles per year became both a linear equation problem and a comment about the destruction of the tropical rain forest.

Transportation provides items such as an actual probability distribution for the number of motor vehicles per household. You do not have to make up a hypothetical one and you, like my students, may be surprised that the city of Providence reports 23% of their households have no cars at all, a group usually forgotten about by local transportation planners with cars. The entire distribution for 0, 1, 2, 3, 4, 5 vehicles is .23, .42, .27, .06, .016, .004 respectively. Transportation also provides such formulas as this formula which I use to illustrate real world functions:

$$C = 0.88 + 27.04/S + 23.874/U$$

where C = annual cost per vehicle mile for a bus system, S = average operating speed, U = peak vehicle utilization. Note what happens as S -> 0 as gridlock develops!

Roundoff can be important! An environmental group commenting on a proposed ozone air quality standard of 80 parts per billion warns not to allow roundoff (to 80ppb) from actual pollution levels up to 85 ppb to meet the standard. In other words, they want the 0 in 80 to be a significant digit so that a reading of 80.5 ppb would be a violation to be addressed. They say "rounding up means people have to breathe more pollution in the air."

This poll result in the *New York Times* can illustrate how newspapers may report the margin of error in surveys: 20% favor reducing spending on the environment, 74% say that is unacceptable. The *Times* explains that in 19 cases out of 20, samples of the size used would result in a margin of error of no more than 3% either way. We can get an interpretation of 95% confidence intervals while indicating something about public support for the environment.

Teaching at a state college where most students are Rhode Islanders leads to looking for a local angle. Perhaps attention to the nearby is a good idea everywhere. Here I note it is easy to get data, graphs, and information on such local environmental issues as the depletion of fish off our coast, pollution of Narragansett Bay, transport of air pollution, etc.

Well, what do students think of all this? I would like to live in a world where my ideas get universally favorable responses (some colleagues seem to live in that kind of world), but I think a more accurate summation is obtained by this quote from our student evaluation forms: "Instructor tried hard to bring interesting side issues into mathematics, sometimes successfully."

By now, my files of such articles are quite large. I would be glad to share any of them with colleagues who write to me at Rhode Island College. I do believe that such files can be used to encourage students to maintain a lifelong interest in paying careful attention to quantitative information, especially with regard to the environment.

Book Review: *Women in Mathematics* by Claudia Henrion

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Women in Mathematics: The Addition of Difference. Claudia Henrion. Indiana University Press, 1997.

NATASHA KEITH'S PERSPECTIVE:

As a high school senior who enjoys mathematics as well as sociology and women's studies, I picked up the book *Women in Mathematics: The Addition of Difference* by Claudia Henrion for my summer reading. The more absolute truth would be that my mother, a mathematics teacher, pushed the book my way. But I was also interested not only because this book appeared more accessible than the dreary statistical literature that I was used to seeing on a family bookshelf, but also because the book discussed a number of stereotypes I recognized and offered case histories of women mathematicians whose appearance in the photographs was intriguing. In other words, this book was humanistic and sociological enough for me to be inviting, and I thought I might even encounter such a book in a women's studies course in the future. Better yet, I might miraculously "find myself," since it has worried me that "UNDECIDED" will be my choice of major on my college application forms.

Each chapter of this book tackles a topic, generally a myth about the role of women in mathematics. Henrion then digests the myth, citing quotations, philosophies, and the case-study accounts of one or two prominent women mathematicians to disprove it, prove it, or prove it partially true. For example, the first chapter, entitled "Rugged Individualism and the Mathematical Marlboro Man" describes the mathematician as explorer, "The image of a mathematician within the mathematics community...is a romantic image of an explorer, living a life filled with adventure, discovery, and excitement." This chapter challenges the idea of mathematicians as "loners," that is, people who work in complete isolation. Here, Karen Uhlenbeck and Marian Pour-El are set as the examples; they describe themselves as loners in long passages of their education, eventually coming to a sense of community in their careers.

The problem for me as a student is that I have never heard of the myth of the mathematician as "Marlboro Man;" I have never thought of the mathematician as the dashing explorer. The book clearly refutes the stereotypes and myths about women mathematicians that are introduced, but this is a somewhat easy task, since, in mathematical vocabulary, any exception to a statement renders it false. But my question about the image of Marlboro man confused me, and made me wonder if the book was really being put forth for readers like me. This kind of confusion persisted through the book, along with more confusion as to whether a stereotype was being broken down or affirmed, found partially true, or side-stepped. In the introduction, the book is described as a study intended to encourage other women to find their own niche in mathematics, to address why women are so under-represented in mathematics, and discover why mathematics still seems to be an isolating field for the most successful women mathematicians. And, while the stories and arguments intrigue me (I discovered mathematicians who were housewives, motherly types, poets, and followers of Zen), I had a nagging feeling that the questions were getting thornier and raising more issues, and that these diverse accounts of women represented a different generation with whom I was not identifying.

With the examples of Karen Uhlenbeck and Marian Pour-El, for instance, I think that the stereotype of the "Marlboro Man" turns into the "Marlboro Woman:" the same individualistic, tough, lonely character, but a woman in the saddle. Uhlenbeck finished high school an independent strong-minded person but with no idea she would do math; she soared in mathematics in college. Marian Pour-El majored in physics at college and ended up as the only woman at Harvard, then worked at Penn while her husband was on the other side of the country. Uhlenbeck says she didn't feet like a mathematician for five years after having her PhD, so strong was her sense of isolation from the mathematics community. (This made me ask what it should feel like to be a mathematician, and what, exactly, a mathematician is.) In terms of tough personalities and sterling achievements, these women are awesome. They are probably the most brilliant of the brilliant. And as a result I found this chapter somewhat intimidating, because I wondered, should one be a Marlboro woman to do mathematics?

But as the book went on, I also found myself surprised and shocked, to the point of disbelief that the 40's through the 70's could have been a time of such profound and awful discrimination. Every woman in this book had experiences from that period and endured discrimination in some form. The stories of the women seem truthful often to the point of pain: they cover a wide range of experiences, from false starts, loneliness and harassment to brilliant successes. There are cheery anecdotes and triumphs, too. Nevertheless, even as these women overcame the problems unfairly thrown at them and rose to prominence, I am not always comfortable with the ways of living they chose. For example, Fan Chung and her husband have turned a wing of their home into a mathematics world—a library devoted to math books and filled with puzzles for their children, who are cheerfully and intensively coached for math leagues—my personal idea of a nightmare. Marian Pour-El claims she didn't receive any help from anyone, and never fought feminist battles, but on the other hand, might she have benefited from the contributions of other women, who had fought their battles? Mary Ellen Rudin is an admirable example of how motherhood and math do mix, and she assumes an almost mythological saintliness, but only because she is willing to fill the stereotype of 'woman first, mathematician second.' Her point to Judy Roitman is: "If you want to help women in mathematics, do mathematics." This is a sort of Booker T. Washington philosophy (a philosophy that is also discussed in the book), entailing that women should patiently and quietly prove their mathematical abilities, and wait for society's trust and recognition to follow gradually. This is not necessarily the injunction my generation has grown up with.

How do women respond to discrimination? The chapter on women and gender politics provides some answers, but creates questions as well. Perhaps this is because I am (as yet) unfamiliar with an environment of "gender politics," or advocating for women's rights. Henrion plans to break down the stereotype that "math and politics don't mix." She asks whether mathematics must be defined as a pure field that denies political involvement, and whether mathematics is an "uncorrupted, uninfluenced, un-gender specific field in terms of the subject itself." She then gives examples of women who are involved in mathematics as well as gender politics. But is it that math and politics that don't mix or is it that mathematicians are not political? Has the mathematics itself changed? Is there a feminist mathematics or not?

One can follow a thread in this book about the individualism of these successful women and the ways in which, as outsiders, they strove to become a part of the mathematical community. But it was difficult for me to see how community works when the evidence is given as personal, individualistic anecdotes, and the actual mathematics is not described. I am unfamiliar with a mathematical community. What is it? Metaphors are provided that liken mathematics to a flower, a sea to be explored, a starry sky, but there is no description of the problems and ideas they worked with. As a result, any mention of a mathematical community seemed more for personal and confidence-boosting needs rather than for dealing with the mathematics itself. Many of the women interviewed concurred that being a women in mathematics often made them feel like the "other," because they were seen first as women and secondly as mathematicians whereas males were "mathematicians." It seems that by isolating each woman in her experience, and not discussing the mathematics in the case studies, Henrion is ironically addressing the women as women-first and mathematicians-second. Though it is doubtful that I would have understood a deeper account of the mathematics, such as the achievements of Fan Chung in discrete mathematics or those of Marian Pour-El in logic, I nevertheless desired it, so that I might understand in what ways these women created some kind of dent in the world of mathematics. It is a little like hearing a news flash that a woman chemist has become a Nobel Laureate, without ever knowing what problems in the chemical analysis she was up against, the plan of her research, and how she found her answer. But perhaps it will never be possible to integrate a discussion of women and mathematics at this level.

The chapter "Double Jeopardy and Race" points at some disunity in the mathematical women's community due to race, and deals more with the racial issues than the gender-related ones. This chapter is in many ways the most engaging and compelling. Examples are provided here of the hard work and the difficulties women overcame; here are stories that are very different from the biographies and conclusions reached earlier. The issues of facing double discrimination, having opportunities denied, the question of attending or teaching at majority black or white schools, and the burdensome sense of having to represent all minority women in mathematics are uniquely analyzed in the cases of Vivienne Malone-Mayes and Fern Hunt.

Finally, in the last chapter, the author completely leaves the subject of women in mathematics to raise the philosophical issue of whether mathematics is based on intuition and ideas or on formality and symbols. She does finally return to the philosophy's effect on women, claiming that women are unable to emphasize the intuitive part of mathematics for fear of being perceived as too feminine. "There is a whole network of associations typically identified with mathematics: rational, objective, a focus on the mind. But this same set of traits is also traditionally identified with men. Moreover, the counterparts of the traits—intuitive, subjective, a focus on the body—are typically identified with women." This seemed to be the most condensed explanation of what Henrion has been investigating all along: why our society has had difficulty admitting women into mathematics, despite their many demonstrations of talent throughout history.

In the end I found myself challenging the challenger, asking: where do the myths themselves come from? Mathematics is a young man's game, math and politics don't mix, mathematics is for white males. A cartoon shows a pregnant woman teacher with the caption, "Somehow she doesn't look like a math professor to me." But what exactly is funny about this? Though Henrion claims that the juxtaposition of roles of a woman and a math professor seems ridiculous enough to be humorous, this stereotype is not in my sphere, and I came away not getting it. This, and other stereotypes that I didn't recognize made me ask: have some stereotypes subsided because of forceful legal action and criticism of the field of mathematics itself, or is it simply that the times are changing, and the whole country is experiencing new enlightenment? The book states that many women are majoring in math but not following through into graduate school or getting employment. There are a variety of explanations, I think, including that these women may be going into teaching or jobs in computer science and engineering, along with men. But is it that progress in correcting the mathematics community is only just beginning at the lower tiers with the new generation? Judy Roitman says that women are turned off from math mainly in high school, because there's the exclusive subculture of the "math nerds," and women aren't admitted. She also says that women don't like to speak up in class because they want to just "let the men talk" while they sit quietly. But this is not part of my experience at all. In the high school and college classes I have taken, the women not only compose the top 5% of grades, they are also the loudest voices and come out with the best ideas. Also, in my high school, most of the top math students are attractive, popular, vocal females, a far cry from the quiet nerd image. But on this account, I can only describe my particular high school.

> "
> ***...have some stereotypes subsided because of forceful legal action and criticism of the field of mathematics itself, or is it simply that the times are changing...***

Certainly the women who are featured here, women who have made it, are a phenomenon. Constantly comparing myself with the biographies of these amazing women was both fascinating and frustrating, Allowing for my personal frustrations, I can also say that reading the book was an adventure into an unexpected world and a fascinating study of character and determination.

By the conclusion of the book, Henrion has explored numerous myths about mathematics. Her delicate treatment and style of writing are simultaneously very warm and analytical. Sometimes the abundance of personal stories makes her generalizations difficult to follow. But the task at hand, of addressing the conflicts within the mathematical community is a formidable one. Hopefully the situation has changed for my generation, but it could probably not improve without the eye-opening accounting in books such as this one.

SANDRA Z. KEITH (HER MOTHER) RESPONDS

When I gave my daughter Claudia Henrion's book, I had hoped she would find something from a woman's viewpoint to love about mathematics, feel reinforced in the idea that mathematics is not just for men, be exposed to some of the roughness of the mathematics world, read stories from women's mouths, and, lastly, come to identify with some role models who could inspire her a little more in the direction of math major than I have been able to do, since mathematics is a field that interests her peripherally. I have been involved all of my mathematical life in attempts to encourage young girls into mathematics, and my daughter was, I confess, something of a test case. Imagine my consternation, then, in finding that she views these women, many of whom represent my own generation, not so much as role models but as historical relics of a time past, a time preposterous, a time to be pitied but shrugged off! There are two points she makes that require a response: the issue of role models and the issue of discrimination, and the two go hand in hand, as Henrion's book neatly makes clear.

In 1988 (Natasha was 7), I directed a conference on Women in Mathematics and the Sciences. Every single woman there had experienced discrimination —and not just the gender harassment that Henrion and her interviewees describe delicately, but more brutal sexual harassment. As difficult as this topic is to discuss even here and now (one would like to relegate it to *New York Times* magazine supplements such as that of June 13, 1999 instead of polluting mathematics journals) this was something that women of my generation dealt with routinely—and the specter of sexual harassment seemed to haunt the halls of mathematics more than other disciplines, although my evidence on this would be anecdotal. It is somewhat surprising to think that the phrase "sexual harassment," with which we are only too familiar, was coined only as recently as the mid-70's (by a group of Cornell University women; the first Supreme Court case on sexual harassment, Meritor Savings Bank v. Vinson, occurred in 1986). Before sexual harassment was a public phenomenon, you simply "handled it." Thus office hours, dissertation advising, and friendly chats with the faculty over sherry might not be for you. That my daughter will probably not experience this sort of negating and isolating experience, that can haunt one for life, is a solid comfort.

However, anyone who thinks discrimination has gone dormant should chat with an Affirmative Action officer; my latest trip to the Affirmative Action Office was a matter of months ago, with a case involving an older student. So how could it be that my daughter finds discrimination exists only now, by reading about it? The only conclusion I can come to that makes any sense, is that when you are 10 or 12 or even 14 or 17, your parents' world is something that doesn't pertain to you. So the education they are passing out now in elementary school is a good thing. Even so, my consternation remains.

Natasha's point about role models, specifically her sensitivity to Henrion's use of the Marlboro Man, recalls a moment of epiphany in my conference. In a session on role models, the speaker was a woman who had successfully been juggling a commute from coast to coast to be with her husband. A woman in the audience spoke up, somewhat aggressively, "How then are you a role model for young girls?" Momentarily we were mute, but it was clear that the point had struck a nerve. Can there be such a thing as too formidable a role model? My conference made abundantly clear that there are organizations that promote healthy youthful role models, but these programs always seem to be budgeted on a string. The millions that may be spent in court rooms over test cases may inspire education on what is politically correct, but women, even of my daughter's generation, will still have a harder time of it, because discrimination breeds and refuses to go away. A month ago a male colleague of mine went to another state to go camping with his adviser to discuss further results to his dissertation. How possible would this have been for me, unless my adviser too, were female? And a class action suit arguing for fairness of salaries is being raised at my university as I write this—I am a poster child for it, having been placed low on the salary scale years ago, and yearly raises determined as percentage increases.

The issue of the Marlboro Man model modulates into the issue of what mathematicians and the mathematics community do, and in this matter I think I agree with Natasha that Henrion's book will require some follow through books. Natasha has completed the first half of calculus at our state university, has had some minimal experience in Math League practicing and taking challenging tests, and her high school math has been dominated by a method of "problem typologies." Thus it isn't surprising that she has had no experience with mathematics as research and discovery. In this learning world, there are no frontiers and no rugged individualists ignoring their neural health as they drive their cattle across mythic continental divides. Even Fan Chung's math league nursery sounds to her like a nightmare. Natasha asks, "what is this mathematical community?" Natasha would recognize that community as teachers. Henrion largely talks about women who are distinguished by "doing" mathematics, and as a result the book did not reach my daughter to the extent I'd hoped, because although the women it profiles had challenges and achievements that she could admire as a young woman, the substance of their experience wasn't hers. In turn, while the book is of enormous significance to me personally and should go into all libraries as an important historical document, it is a pity there isn't more on mathematics teaching, because teaching is the bridge for young girls.

(Incidentally, the cartoon which my daughter did not find funny was one I suggested for the MAA book, *Winning Women into Mathematics*.)

A Tribute to Ramanujan

Mahesh Dube
Indore, India

Amidst the Southern hills
Of an ancient land of myths,
Where Nature has a wild aroma
Komala carried within her
drops of heavenly nectar—
To nourish the blessings of Goddess,
And a mother gifted a blossomed mind
    to the mother Earth.
The omniscient scholar of every integer,
Oh! Ramanujan, the mystic and the seer.
To the masters of the field thou became—
    the child dearest
Pride and glory of history and
    privilege of present.
Moving with an inner light,
Through the labyrinth of numerals
intricate of abstractions were tamed into
Raptures of sweet rhythms!
Charmed and exalted by the music of
    numbers and functional oscillations
Dreams cast on thee a spell of
    Sums and integrations.
Generously but shyly did thee disclose—
A circle trick and modular stroke.
Didn't thy Tau-functions provoke
In the pages of Mathematical Society, London
    Several congruent relations?
And bring home many conjectures
with Littlewood, Hardy, Watson and Rogers.
From frational ecstasy or Mock-Theta agony
    Of thy notebook spring
Beacons of our mechanics, cosmology
    and super-string.
But alas! the Zenith of thy knowledge
Became the nadir of thy physique,
And from the banks of Cauvery
Across the western horizons
Thou immortal one leaving the fragrance
    With Jankee
We bow to thee, oh! Ramanujan—
    We bow to thee.

Use Your Head: Mathematics As Therapy

Miriam Lipschutz-Yevick
New York City

Both of my parents lived into their mid-nineties.

My father suffered a mild stroke at the age of ninety-one. "She is a mathematics professor" were the first words he uttered to the nurse, proudly pointing in my direction upon regaining his speech. I took it upon myself to give him biweekly speech and arithmetic lessons to counteract his aphasia. I would ask him to add, subtract and multiply orally. His answers came rapidly when produced automatically but with much greater mental effort when he slowed down to think them through. Yet he valued these lessons and stayed with them to exhaustion. He was in touch with skills embedded in his head since early childhood. His mind had been reactivated. Some nine years later when my once brilliant mother's strength was fading, I too would sit by her and show her the wooden numbers of a child's puzzle. "What number is this?" "Six," she would whisper. "And how much is six plus six?" The right answer came from her struggling lips as I squeezed her blotched and gnarled hand to express my pride. Her eyes lit up knowing that she still could think correctly.

My parents enrolled me as a child in one of the Netherlands' first Montessori schools. We frequently started the day gathered around the teacher, who fired arithmetic problems at us and made us solve them mentally. With our minds revved up after one half hour of this activity, we would get to our work tables to engage in our individual projects. I often entertained myself in bed, unable to sleep during the long summer dusks, with proposing arithmetic problems to myself in my head and enjoying the patterns I would discover in their answers.

These mental gymnastics stood me in good stead when, upon arriving in the U.S. at the war's outbreak, I applied for a scholarship at the Lyceé Français in New York so as to maintain my educational headstart over American high schools. The mathematics teacher who tested me in an oral examination asked me to consider some algebra problem and to explain my thinking aloud. "Always think things through mentally first," he said, "then you will see the problem as a whole." I have taken his suggestion to heart ever since. I have seen the outlines of a solution to a research question emerge as I let it wander in my head before falling asleep. "Think of your math problem when scrubbing a floor, or when the kids are screaming, or when in bed without your spouse." I used to say to my adult evening students, "It will take your mind off your other problems as well."

I developed a special course for these students entitled *Mathematics for Life and Society*. As a retired professor I presently am teaching this course to residents of the assisted care unit at the Windrows in Princeton, N.J. My students of all ages past 70 are rediscovering the basic math they thought they never were good at as I illustrate its relevance to our present world and life environment. I encourage them to think the questions through and to seek the answers in their heads. I have been greatly gratified to recognize the alertness and self-confidence (as well as the pleasure) among my audience due to this mental reawakening.

My grandchildren, unfortunately, were taught to use their fingers when adding and subtracting. They graduated from counting fingers to calculators and hence, I presume, they will move on to computers. They never developed a sense of the combinations of magnitudes which are revealed when seeing the abstract number patterns in the head.. Where will they repair to when the years lie heavy upon them and the clutter of too many stored details and similitudes of past events confuses the mind? For they will be lacking this restorative haven of precision, economy and logical thought bestowed on us with our first fundamental knowing of numbers.

Coherence in Theories Relating Mathematics and Language

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ABSTRACT
Various relations between direct and metalevel studies of language and mathematics are examined from an interdisciplinary perspective, in order to sort out in what way these interactions may be seen as, or lead to form, a coherent pattern of ideas. After setting up a rather precise map of this interaction, it is argued that coherence is necessary besides being possible.

1. INTRODUCTION AND BACKGROUND

Relations between language and mathematics are, in a broad sense, increasingly in the focus of attention of recent studies in the philosophy of mathematics as well as in mathematics education.

In the philosophy of mathematics, two current tendencies may be identified. Both are rooted in the basic idea that mathematics represents a special form of language use, according to a more general theory on how language use is constitutive for structures of abstract meaning, represented by figures as different as Peirce, Wittgenstein and Vygotsky. The first class of developments of this idea is diachronic in nature, as it is based on what is claimed to be historical evidence for the "dialectical nature" of the special language use involved in the construction of mathematical meaning; furthermore, it insists that this dialectics can only be understood as socially situated. At times, this direction approaches what might rather be called a "sociology of mathematics." Considering it a philosophy of mathematics, the basic claim seems to be that the nature of mathematical knowledge can only be studied indirectly, mainly through the institutions, interpersonal relations, etc., that are related to the creation and dissemination of mathematical knowledge. Representative accounts are found in (Hersh, 1979), (Kitcher, 1984), (Tymoczko, 1986), and (Ernest, 1998), where the philosophical nature of this viewpoint is also explicitly defended. One characteristic feature of such an account is that it is deductive, in the sense that it infers from general beliefs (such as "knowledge is socially situated") and related theories (notably sociology and elements of sociolinguistics), to the special case of mathematics, accommodating the specific characteristics of this special case in the general picture to the extent such specifics are considered at all. In fact, reference to actual mathematical practice is typically scarce. Even in (Ernest, 1998), the main such reference is indirect, namely via Lakatos' classical study (1976) of the history of Euler's polyhedron formula.

The second class of developments is focused on structural relations (similarities, differences, dependencies, etc.) between linguistic and mathematical knowledge, taking as a basis a synchronic, interdisciplinary analysis of actual mathematical and linguistic structures. The objective of such studies, then, is to shed light on the nature of mathematical knowledge through its relations with the (more "commonplace," but clearly not completely understood) nature of linguistic knowledge. It seems fair to say that there are more claims of such structural relations (which may be found scattered throughout the literature) than actual, detailed studies based on linguistic methods and explicit mathematical content, but at least, we have presently several examples, and some tentative theoretical frameworks, cf. (Halliday, 1974), (Pimm, 1987), (Rotman, 1988), (Walkerdine, 1990), (Winslow, 1998). Any such theory will, in principle, be inductive in nature, proceeding from analysis (in a more or less formal sense) of actual texts (in a broad sense) to general hypothesis and claims about patterns and characteristics of mathematical language use. The perspective of the present paper falls mainly in this second class.

It seems clear, even from this quick sketch, that the two viewpoints are not theoretically opposed to each other; rather does the difference reflect incommensurable views of what constitutes (or perhaps, what is important in) a theory of knowledge.

Turning now towards mathematics education, the main theme has been the roles and functions of natural language in the learning of mathematics, especially the mechanisms and the learning potential of discursive interaction in mathematics classrooms and correlations between performance in mathematics learning and competency in the language of instruction. It seems clear that such roles, functions and correlations should be both relevant to, and enlightened by, the philosophical issues alluded to above. The fact is that such relations are not studied in depth, especially as concerns the second of the above-mentioned viewpoints on mathematical knowledge.

The aim of this note is to argue that this relation may be established in a coherent way, while taking seriously into account both mathematics (the centerpiece, after all!), linguistics (not just folklore beliefs about "the social nature" of language), the philosophy of mathematics (as a discipline with certain basic issues and a long history related to philosophy in general), and mathematics education (as a young and quite vital discipline addressing highly important problems for education in general). As intermediate steps, we need furthermore to accommodate certain areas of sociolinguistics and language pedagogy, which are associated to (and partially derivative of) linguistics as such.

The main point of our discussion is that leaving out the full perspective of one of these disciplines could (and does in some cases) mean turning from "communication oriented theory of mathematical knowledge and learning" to either a very narrow study of two-sided correlations, or (in the worst case) to incoherent, unprofessional gossip.

The reader may wish to consult, at this stage, the figure in Sec. 7, in order to see at once the coherence of the first six sections, despite their apparent disparity. Moreover, the reader only interested in metamathematics (not in learning theory) could omit Sec. 5 and 6, corresponding to the right third of the mentioned figure.

2. FROM MATHEMATICS TO LINGUISTICS

This is the first and easy part of the story, as it is essentially just to be recalled (not constructed anew); the way back, from linguistics to (meta)mathematics, is the difficult part.

The flow of ideas from mathematics to linguistics can be regarded at two levels, which, after the early days (Harris, 1970), seem to be increasingly distinct in practice. The first is similar to the use of mathematical models in natural science fields, and is the source of what is called mathematical linguistics. Here, one studies certain mathematical systems which are, at the outset, defined in order to model certain aspects of natural language (typically within syntax). The problem vis à vis linguistics seems to be that the study of such models quickly generates interesting questions and sophisticated techniques, while "applications" (insights about natural language) become somewhat secondary. This is of course also the case with several other "mathematical" disciplines, such as mathematical physics or mathematical biology. For a readable introduction to mathematical linguistics, see e.g. (Gross, 1972).

The second level of influence is less direct in nature. It can be described as a transfer of methodological approach (rather than concrete theory), in this case resulting in a shift from the traditional study of "local phenomena" (e.g. the historic development of a certain word or expression) to the study of language as structures which are amenable to "global" study, much like (but not as) a mathematical structure. Although apparently this is a weaker kind of impact, it has been much more important: first of all, it is crucial in the rise of modern, structural linguistics. It has also, at times, been significantly inspired by results and methods from mathematical linguistics, as in Chomsky's famous argument that language structure cannot be generated from finite evidence alone (Chomsky, 1957, 83).

It goes without saying that the transfer of research methods among fields which were, traditionally, as separate as mathematics and linguistics, is neither harmless nor exempt from controversy. Indeed, it is highly questionable whether the study of language can address its most interesting aspects with a rigid and formal approach inspired by logic and mathematics (cf. also Sec. 2); this is, by the way, not the claim of structural linguistics as such. On the other hand, such an approach has not only given a new and better understanding of syntax and other parts of formal language structure, it has also added a dimension to our understanding of what research in linguistics and more generally in the humanities may be.

Incidentally, this second level of influence is not restricted to the field of linguistics. In fact, structuralism, as a general trend in the human sciences, may to a large extent be regarded as the result of this transfer of systematic approach from mathematics, see e.g. (Piaget, 1968) and (Gibson, 1984, B1).

3. LINGUISTICS BROADENING: SOCIOLINGUISTICS AND LANGUAGE PEDAGOGY

Sociolinguistics may be roughly defined as certain domains of inquiry in which the study area and methods of linguistics are extended to embrace wider areas than the structure of natural language at phrase level. These extensions happen in different dimensions: to cover natural language use at text level (discourse analysis), to analyze human sign systems beyond written and spoken natural language (semiology), and to study the use of natural language within various segments of a human society ("sociolinguistics" in a narrow sense). A well-developed area, which has affinities with all these aspects of sociolinguistics as well as with education, is the field of language pedagogy, in which the subject of study is the teaching and learning of natural language (mother tongue or foreign), cf. (Stern, 1983) and (Stubbs, 1986), that contain also references to work done in sociolinguistics. A main fact to observe here is that all of these extensions have significant roots in structural linguistics and directly or through structural linguistics in the Saussurian view of language as a sign system. In particular, they all carry the imprint of structuralism, and, with it, of a more or less direct transfer of methodology from mathematics. Discourse analysis, being a straightforward extension of grammar in the traditional sense, clearly confirms this claim, although the structural description may (and does) in this case exhibit a more pragmatic character. The idea of sign systems, in which signifieds are more often than not represented by signifiers of other signs, may be regarded as a mathematical metaphor from the structuralist view of mathematics as concerned with abstract "relation patterns" (Resnik, 1997). The study of language use which is particular for certain societal groups may look, at first sight, less subject to such relations, yet its inevitable need for describing these particularities (e.g. in terms of syntax or lexicology) ultimately forces it to interact with structural categories of natural language (Hudson, 1986). Similar remarks apply to the area of language pedagogy, where the influence of cognitive psychology (e.g. in the Piagetian sense) only adds further imprint of structuralist approach (in the sense outlined in the introduction).

4. RELATIONS BETWEEN MATHEMATICS AND KNOWLEDGE ABOUT IT

After having outlined the flow of ideas from mathematics as far as to language pedagogy, we return to mathematics and begin to look at it from outside; theories about the nature and structure of mathematics as a whole (assuming this "whole" makes sense in an appropriately delimited sense) will be said to belong to the domain of metamathematics. In particular, this includes theories which are (at least similar to) mathematical theories themselves, such as various forms of logic, or category theory. It also includes theories of philosophical nature, to the extent these are concerned with mathematical knowledge. In fact, one has yet to see an interesting theory on the foundation of mathematics which does not raise philosophical problems.

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A main torment of such theories seems to be that a credible solution to one problem often seems to have unacceptable consequences for some of the others...

The issues considered here thus ranges from the technical foundations of the subject, over its functions in other fields, to ontic and epistemic questions. A main torment of such theories seems to be that a credible solution to one problem often seems to have unacceptable consequences for some of the others; for instance, the various brands of realism offer coherent ontologies but also harsh difficulties with epistemic questions, while the theories that regard mathematics as a social entity seem to be more coherent on epistemic issues than as regards foundations. What I would like to do here is to replace the abstract subject (mathematics) with its appearance in the world, which is plainly textual (communication about, or using, mathematics). It appears that many controversial issues, especially as regards ontology and foundations, arise from the trying to grasp "the thing behind," the silent, unarticulated Mathematics behind its manifestations. In this sense, what we are left with could be said to be of "social" or "human" nature, just as other communicated realms; but it seems to me that to accept this is an empty triviality that is controversial only to the extent one has not abandoned the idea of finding mathematics elsewhere than through mathematics as encountered in the world. Whether or not there is such an out-wor(l)dly thing, I think it is legitimate and even wise to restrict rational inquiry to its complement.

Settling with this, we have excluded from our perspective of metamathematics only certain forms of realism which claim that mathematical objects have existence beyond their presence in communication about mathematics. Notice that this does not necessarily exclude them from being permanent, and, as far as we can see, even eternal; the universal grammar of human languages is a perfectly assuring analogy in this respect. On the contrary, having to rely on invariant properties of a subject which is not explicit in its articulated forms, seems to me to be of rather little assurance as regards the stability of the subject. We are thus left with all foundationalist theories of this century, and almost all of the relevant analytic and continental philosophies. Mathematics itself has limited use in judging between them. Indeed, we know at least in a certain limited sense (from Gödel's work) that no foundational theory will be "proved" mathematically. However, mathematics is in various forms is taken into account in each of them although not in all of them in a way consistent with the picture we are about to set up. For instance, Brouwer's "first act of intuitionism" (intuitionistic mathematics is an essentially languageless activity of the mind, in Brouwer, 1981) suggests that this school may fall partially outside the perspective of this paper, but the main reason is its prescriptive nature, as discussed in the next section.

5. LINGUISTICS AND METAMATHEMATICS

Metamathematics is inspired from linguistics in narrow and in broader ways. In the most literal sense, the logico-foundational studies of Russell, Gödel, Tarski and others are certainly a kind of linguistic inquiry into formal (mathematical) language in a very special technical sense. I claim that these studies have really little to do with linguistics and mathematics in the plain sense. As concerns linguistics, only few, especially Montague (1974), have tried to connect theories of formal languages directly to linguistics in the usual sense (dealing with natural language), and it seems to have little or no direct impact on modern linguistics. Regarding mathematics, the problem is that the formal languages analyzed in this tradition are quite different from the way mathematics is communicated by mathematicians other than these logicians. Rather than being a descriptive (or naturalistic) account and analysis of mathematical language use, this field constructs, to a large extent, the language it analyzes, although with the intention of creating a more consistent form in which to formulate existing mathematics. It thus has a prescriptive element which is quite foreign to linguistic methodology; right or wrong in language use is relative (to the grammar etc. of a speech community), not a matter of legislation. No linguistic theory of natural language (not even that of Montague) would start by constructing alternative forms of the language in order to put our understanding of the less perfect original on firm grounds.

The broadened scope of linguistics (Sec. 2), especially semiology and discourse analysis, offers various means of interpreting mathematics as language use. The "semiotics of mathematics" was opened, as a new perspective, in (Rotman, 1988), while ideas from discourse analysis have been influential in recent work in mathematics education (e.g., Pimm, 1994 and Sfard, 1998). The major question is now: which theories, among the fairly large and varied assortment available, are more likely to provide us with tools to make this interpretation faithful? One popular notion in this respect has recently been that of metaphor (e.g. Sfard, 1994, Lakoff-N'Òez, 1997), a notion which otherwise is especially used in the analysis of literary texts. While this has led to insights regarding the construction of meaning in mathematical texts as well, it rests purely semantic; it establishes an analogy between the meaning construction of literary and mathematical text, but it does not in itself enable contact with the specifics (or, more plainly, the obvious differences) between these. In short, the notion of metaphor would be much more powerful if associated with a firm theory of the linguistic specifics of such texts. Furthermore, the field of structural linguistics seems a quite obvious place to start, given its resonance with mathematics. This certainly does not imply that we are to restrict our investigation of mathematical text to their form but that our main goal, to describe the nature of their meanings, is better served when enlightened by a natural theory of the nature of their form. I believe we are here at a crucial intersection of the argument;

the question is, why not just bypass all this formal business and stick with pragmatic analysis of how thoughts are exchanged, regardless of well-formedness? Because in doing so, we bypass what makes mathematical communication both special and possible. The point is made as follows, in the context of grammar in natural language, by Moravcsik (1976): There is no thinking without rules. Both logic and grammar conspire to make it possible for us to articulate thoughts. It is absurd to think of rules as restricting thinking; rules of coherence, consistency, and grammaticality are what makes thinking possible. An analogy with games like chess should make this clear...The very possibility of playing chess is given by the fact that the game is defined in terms of rules...Learning and following rules...enhance our lives and enable us to be free to participate in a large number of activities.

Indeed, it is obvious that mathematical texts conform to strict formal rules in the sense of structural grammar: apart from symbol strings, the text consists of phrases of natural language which are constructed more or less as in other natural language texts; the symbolic parts are also highly regulated, although not according to usual linguistic principles. The interplay between symbol language and natural language is complex but clearly also crucial to the meaning of the text both at sentence level and at discourse level. A systematic yet still somewhat sketchy description of this interplay, from the point of view of structural grammar and discourse analysis, is given in (Winslow, 1998, Sec. 3) and (Winslow, 1999, Sec. 6). The latter reference also discusses the relation to Chomsky's view of linguistic knowledge. We can summarize the main results as follows:

Mathematical texts contain a certain regulated mixture of natural language and symbol languages (the latter including figures of geometric nature). At the first level, the syntax of the text is that of natural language, with certain phrase elements (e.g., a noun-phrase) replaced by symbol strings (e.g., an equation). Then, at the second level, the symbol strings have their own (universal and context-bound) syntax, which interacts systematically with the natural language syntax of the first level, e.g. in regulating "replacement."

We may describe the communicated realm of mathematical texts by the specific language use they represent, i.e. as a family of linguistic registers. This description includes the syntactic observations mentioned above, but also certain patterns of discursive practice which are closely linked to the syntactic phenomenon of transformations. An analytic tool in the analysis of discourse in mathematical registers is the notion of ensemble, which is a dynamic structure of textual information around which the discourse is centered.

As in the case of natural language, the complexity of mathematical language use forces us to accept that human competency in this domain has a non-void (innate) "initial state."

Metamathematical questions, delimited as in this section, address the nature of mathematical knowledge as evidenced by performance, i.e. from communication among human beings; this knowledge consists roughly of communicating competency (analogous to knowledge of a natural language) and factual knowledge (a finite number of sentences believed on linguistic evidence to be true). Notice that it is the latter, finitary part of mathematical knowledge which has often been in focus of philosophical studies; for us, they are the trivial part, and may in principle be considered lexical material. None of the traditional problems are swept under the carpet this way, because the evidence underlying this material, as well as the individual's belief of it, is highly dependent on communicative competency. Yet there is a shift of emphasis, well in line with mathematical practice: understanding a proof is mainly a question of realizing certain transformations as acceptable to the grammar of mathematical language use. The context is built up using, but not itself constituting, the register. Thus, as with natural language, the (extended) lexicon is occasionally revised during discourse practise. The elementary logical basis of mathematical reasoning another traditional source of controversy is then viewed as an integrated part of human language capacity, much in the Wittgensteinian

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Understanding a proof is mainly a question of realizing certain transformations as acceptable to the grammar of mathematical language use.

sense of rule following as a condition to engage in meaningful language acts.

6. RELATIONS BETWEEN THE KNOWING AND THE TEAMING OF MATHEMATICS

In any domain, theories of knowledge and theories of learning interact. In everyday language, learning is a process towards or between states of knowing; a (problematic) mathematical metaphor could be a function of time (representing knowledge) and its derivative (representing learning). The problem with teaming mathematics (and language, an analogy to which we return in Sec. 6) is that it is not an accumulative process which is easy to test in the function picture; we are talking of a function with no obvious range and means for evaluation (this, then, applies to the derivative as well!), and it is definitely not a process which may reach a final "perfect" state. Apart from the most trivial kinds of recitative knowledge, such as knowing the multiplication table from 2 to 10, it does not really help to restrict or specify our attention to certain domains of mathematical knowledge.

It is also clear that there is no easy relation between metamathematics and mathematics education, understood as the theory and practice of learning and teaching mathematics. As Ernest (1991) rightly points out, we are talking here of resonances rather than logical implications between theories of knowledge and learning. In particular, the function image is also false in suggesting that the latter are derivatives of the first in any sense. All too often, educators have been more or less explicitly drawing links between false dichotomies of (what they conceive of as) "good" vs. "bad" views of knowing and learning mathematics, often as a circular justification of the "good" views.

How are we to proceed, if we are nevertheless sustaining that substantial links exist? I believe a key is to realize that, in fact, knowing and learning mathematics are not separable phenomena; acceptable signs that an individual "knows" mathematics (his "performance") will in all interesting cases consist in the production rather than reproduction of mathematical text, hence involve an element of self-induced learning on the part of the knowing individual. To see how a disciplinary distinction may then be sustained, we have to further graduate the rough definition of mathematics education given above, to comprise at least three main layers, each with plenty of room between them:

a) Mathematics acquisition (theories about the universal aspects of individual mathematics learning)

b) Didactics of mathematics (inquiry into how individuals may support learning of other individuals)

c) Methodology of mathematics teaching (concrete methods of didactics implementation in a specific context of contents, age group etc.).

It is clear that these three are also closely interacting in practice, but at least the traditional tension between theory (ranging from philosophy to psychology) and implementation (such as classroom teaching) can now be located as the span from a) to c), the former being the main site of interaction with metamathematics. In (Winslow, 1999) I have given an example of how metamathematics and a) interact, in particular how Chomsky's version of Plato's problem arises and may be approached in ways similar to what is found in linguistics models (of initial states and so on).

7. SOCIOLINGUISTICS, LANGUAGE PEDAGOGY, AND MATHEMATICS EDUCATION

There are two senses in which language theory may occur in mathematics education research. The first is related to natural language use in mathematics teaching, the second to mathematical language use as a central aim, and often also difficulty, of learning. In classroom discourse, one may typically find a maze of intertwining phenomena related to these two categories, but it is seems clear to me that to study the first without the second (that is, completely neglecting the linguistic specifics of the subject matter taught) is not a task for mathematics education, but rather for general educational discourse analysis. By now, there is an abundance of studies (not to speak of collected data) regarding communication in mathematics classrooms, and I think it is fair to say that few of these are explicitly grounded in coherent theories of both mathematical and natural language discourse. Because of the complexity of mathematics classroom discourse, in particular the delicate mixture of registers, only parts of which are mathematical, this leads to a situation where the results of research become incommensurable analyses of special cases, with reproducibility in other contexts far out of sight. Notice that this is not meant as a critique or denial of the value of those works; indeed, in a pre-paradigmatic phase, such a situation is an inevitable and even necessary step towards the formation of coherent research programs.

A perspective suggested by our discussion is the potential parallels between mathematics education and language education, and furthermore the necessity of making this explicit has so far been almost completely ignored in the study of linguistic aspects of mathematics education. The following are quite obvious domains of inquiry in which this parallel can be pursued:

How does knowledge of grammar behind "known language" uses (natural language mother tongue as well as known mathematics) affect learning of the new "grammar" (of mathematical language use)? Research partially along this line may be found e.g. in (UNESCO, 1974) and (Saxe, 1988).

In what ways does mathematical discourse competency develop from other forms of discourse in mathematics learning environments? See e.g. (Pimm, 1994) and (Sfard, 1998).

What are the roles of "learner factors" (such as affective factors, maturity factors and aptitude, cf. (Stern, 1983)), as studied in language education, in the learning of mathematical language use? This parallel is discussed e.g. in (Winslow, 1998, Sec. 4.3.2) and, at a much larger scale, in (Clarkson and Ellerton, 1996).

What diagnostic teaching forms (including tests) can be used to address particular language-related troubles in the learning process? This issue also has affinities with recent neuro-psychological research on the cognitive structures behind linguistic and numerical activity (Dehaene, 1997).

Are there mathematics-specific language disabilities, and how are they related to the classical types of (natural) language disorder? Research along this line may be found e.g. in (Donlan and Hutt, 1991), cf. also (Stubbs, 1986, Chap. 10) for background on "conversation disorders."

For all of these (and similar) points, it should be acknowledged that additional complexity (and, presumably, difficulty) arises in learning which is meant to include mixed discourse abilities, that is, the understanding and production of text in which mathematical language use occurs in contexts other than "pure mathematics," that is, in applied contexts. This, on the other hand, also parallels obvious issues in foreign language teaching, which is seldomly restricted to technical acquisition of the language (but includes also e.g. cultural and literary elements).

8. THE GRAND PICTURE AND THE NEED FOR IT

Viewing the preceding sections separately, I hope to have given the reader an impression of the deep links between the study of natural language structure on the one side, and the study of mathematics as a domain of knowledge on the other side. Both come in three more or less consecutive layers: the field itself, its metaaspects and its learning. Putting them together and drawing the flows of intellectual current which were described, we arrive at the diagram of Figure 1. It is important to note that not all arrows have the same status at present; especially the downward arrows are only now appearing in tentative ways, and as mentioned in the introduction, these ways are partially incompatible if not incommensurable.

The question is now: to what extent do these flows add up to a coherent picture? And, even more importantly: to which extent could and should they? By coherence, I do not mean strict commutativity in the

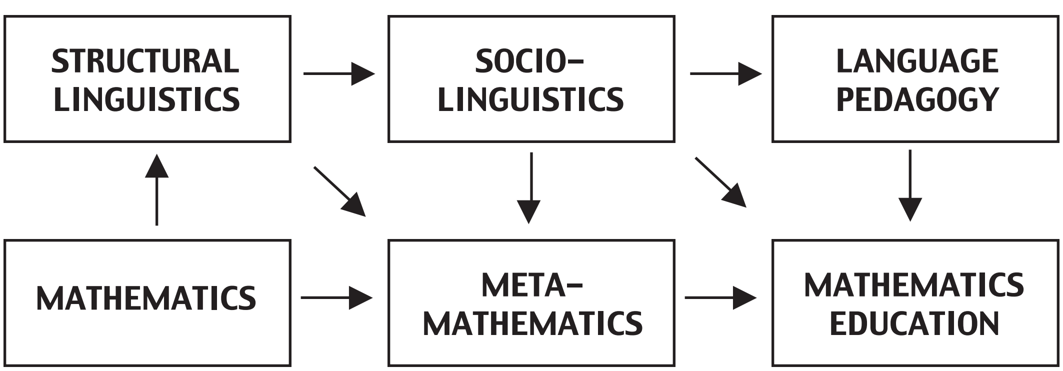

Figure 1

mathematical (mapping) sense, but rather that the flows are conceived as coordinated around each of the four triangles, representing (from the left): the influence of mathematics on metamathematics and linguistics, metatheories inspired by linguistics, the uses of sociolinguist perspectives in metamathematics and mathematics education, and the (emanating) flow of ideas from sociolinguistics through language pedagogy to the theory and practice of mathematics education.

I will close this note by sketching my main arguments that coherence is crucial for these interactions to play a significant role in metamathematics and mathematics education. First, assuming that our inquiry addresses issues (epistemology, foundations, etc.) which are specific to mathematics, it is evidently important that mathematics is in a substantial way taken into consideration. If we are to make the case that language type phenomena are, in a non-superficial way, crucial for these issues, then mathematics itself must be accommodated in our framework from the outset. This particularly concerns the left triangle of Figure 1, without which the rest has no direct link back to mathematics (this is to a large extent the case in much of the current relevant literature).

A crucial issue is the role of conversation in the learning and creation of mathematical knowledge (cf. e.g. Ernest, 1998). Here, of course, mathematics is taken into consideration, but usually only through (a few) specific historical instances. This is highly unsatisfactory if one is interested in the general case, but in the absence of a coherent and general linguistic theory of mathematical language use, there is no better option. This is why we are forced to substantially involve the apparatus of modem linguistics. Interdisciplinarity is obviously a necessity for the study of relations between as disparate domains of inquiry as dealt with here. Because of the traditions of professional training, few if any agents in the research communities will be experts in each of the six participating fields. The result of having no consensus or common overall perspective may be a wealth of bidisciplinary efforts, in which important input from other parts of the pattern is ignored.

On the other hand, a coherent understanding of interdisciplinarity in this area may lead to a new type of professionalism. This phenomenon is well known for bidisciplinary work, as the examples discussed in Sec. 1. My vision for the linguistic study of aspects of mathematics is exactly that this could be the case for the six-discipline interaction mapped out in Fig. 1, engaging mathematicians, linguists, philosophers of mathematics etc. and particularly "combinations" thereof in an explicitly articulated enterprise. If language use in mathematics is subject to defining rules of linguistic nature, then this enterprise is essential for our understanding and dissemination of mathematical knowledge.

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"As far as the laws of mathematics refer to reality, they are not certain; as far as they are certain, they do not refer to reality."

--Albert Einstein

Operationalizing Interactive Learning Paradigms Through Cooperative Learning Activities 100% of the Time In Math Classes

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An interesting hypothesis arises during discussions on teaching techniques used in mathematics classes when teachers compare lecturing versus cooperative learning. Some people postulate that it is necessary to present information to students before they attempt to understand it themselves. I believe this supposition clearly articulates the difference between the two paradigms. Cooperative learning sets very high expectations, that the students can understand the content by taking responsibility for their learning, versus the instructor assuming she/he must cover the material for the students first. In the processes described below, cooperative learning is used 100% of the time in class, thus establishing that the students can indeed learn mathematics with minimal intervention from the instructor. Students rise to the occasion and often exceed their own expectations when they work cooperatively with their peers.

Lecturers maintain that students initially must see a presentation of content material from the professor before they start the process of internalizing the concepts. They claim that students have to see examples of someone working out math problems or constructing computer programs or even solving word problems to begin to understand the underlying concepts. The presentation doesn't have to be long, but it must be there. For students who are good enough to learn the rudiments from textbooks the presentation step can be skipped, they maintain, but most students need a live presentation. Math is something that is better presented live because the students will be able to see the stages of a derivation much better than can be presented in a book.

The argument made above is a common assumption made by many teachers today. The following description of my class procedures, using cooperative learning, is intended to demonstrate that lecturing in math classes is not necessary. Instead a process is used which facilitates student learning by encouraging them to try to understand the material on their own, first by reading the text and then by working out problems together with their peers, finally with the teacher intervening only when absolutely necessary.

In order to set the tone of the class I send my students a letter prior to the beginning of the semester which includes a humorous introduction to the class and cooperative learning, a course syllabus, and a writing assignment in the form of a math autobiography. Students are asked to read the first chapter and start working on the text problems. The first chapter includes review materials from the prerequisite course. My intent is to emphasize their responsibility in the learning process well before the class starts and to demonstrate my own interest in helping them become independent math learners while providing a strong and varied support system in and out of class.

Students are asked to read the text before class and are provided with a class syllabus specifying exactly which sections they are responsible for on a given day. This may be revised during the semester as the class progresses. Students are also asked to complete as many problems as possible prior to class. (They have student manuals which provide worked out solutions for all the odd problems in addition to the text examples). About half the members of each class actually do the work prior to class.

At the beginning of class, worksheets are handed out which contain problems or questions which cover the day's content. No lecture is given until after the work is completed and then only if absolutely necessary. The problems progress from simpler problems to more complex. The students work in pairs or larger groups, usually with 4 people to a table. Sometimes problems are worked out on the board by students who explain and defend their solutions to the whole class, or students work directly out of the text together. We have a workbook form of text in the math classes which encourages students to write in the book. By solving problems first in groups, the students are more likely to volunteer to explain their solutions to the whole class. The strategy of starting with the simpler problem is designed to guarantee that students will be able to successfully complete the initial part of each assignment. If they need help, they are encouraged as a group to go back to the text to obtain examples of more complex problems.

I circulate around the class, observing each group's progress and making suggestions about how they might go about finding the answers to their questions. Initially I do not answer questions directly. The students are encouraged to use their text and any other student in the class as resources. Those who did not do the reading and practice beforehand have an opportunity to do so at this early point in the class.

If enough students appear to be having difficulty or generally are making fundamental mistakes, volunteers will be asked to put their solutions on the board to provide a basis for discussion. This might be considered "showing them" what to do, but the advantage is that the explanation comes from the students, not the teacher imposing a solution on them.

The students then go back to work and try to resolve their questions. If they are still confused, I will then facilitate a whole class discussion and try to elicit the source of their confusion. The focus is still on the students, not on me as the person who can solve all their problems and explain everything to their satisfaction.

Group quizzes are often used as a form of review after we have covered several sections within a chapter. First students work individually and then they compare answers and try to reach agreement on the final answer. At this point it becomes clear which students are competent and which are not, and I can encourage those who need extra help to obtain tutoring outside of class. On occasion I have postponed tests because I have observed enough unprepared students to know that a test would be a disaster. Coddling? I do not think so. Sometimes, with all the pressures students are under today, there is a critical mass that just aren't ready to demonstrate their knowledge through a test at a time specified for our convenience. That is not to say that my courses are open ended. They are not, but within a syllabus there is some scheduling flexibility, which is appreciated by the students.

Finally, an in class test is completed individually by each student to maintain their accountability. A mastery approach is used where students have an opportunity to correct their mistakes during the exam, before a final grade is calculated. Here again I walk around the room observing students' progress. When they complete their test, it is checked immediately and any incorrect answers are circled, without indicating what mistake was made. Students then have an opportunity to make corrections. If they get below an 80% after corrections, then they need to take a new test outside of class.

Every step of this process is intended to encourage the students to take responsibility for their learning. This sets very high expectations for the students and myself as the facilitator. My role is to provide materials which will help guide them through the process and work with them to develop appropriate group interaction skills, which are sorely lacking these days. I am intensely involved in each class as I circulate and talk to students individually or in pairs or groups and guide the classes between whole group discussions and individual work.

There are other cooperative processes, such as jig saws, math olympics, make up your own tests, pair reading and writing, group reviews, etc., which are used in addition to the one described above so that the classes never become completely routine. Student responses are that the classes fly by, and they are exhausted at the end of class but feel good about what they have accomplished. By the end of the semester the better students have learned how to become more independent learners, their math phobia has all but disappeared and they actually begin to like math, and the less motivated students have learned more math than they ever expected. In class the students cover more material than I could ever hope to lecture on and obtain their understanding. And, they understand in a way that makes sense to them because they are developing their own solutions.

My classes generally run around 25 students, but I have done this with classes as large as 50 and adult groups of 100 in seminars. Obviously, the larger the class, the harder it is to personalize it. The above procedure would need to be significantly modified for larger classes through the use of in class TA's and other mechanisms. A class of 500 would be very questionable. CL is not meant as a cure all for economic problems and solutions imposed by administrators. It is well established that smaller classes are better pedagogically.

The procedures described above have evolved over a long period of time through a process of trial and error. It not recommended that new teachers initiate this extensive a cooperative learning system without first participating in training programs and conferences dealing with cooperative learning techniques. It takes time for teachers to develop a comfort level and develop a degree of confidence with cooperative processes. A good approach to incorporating CL in math classes would be to initiate one or two new techniques each semester until a full repertoire of activities is available to chose from.

The Need for Interviews in the Mathematics Classroom

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"God does not care about our mathematical difficulties. He integrates empirically."

--Albert Einstein

Mathematics and Sex

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What makes mathematics so appealing even to those who have not been drugged with a dose of higher mathematics is perhaps its intimate relationship with other disciplines. Most expository mathematics books targeted at the educated public have not failed to lure us into the cold beauty of mathematics.

Be it *mathematics and art*, or *mathematics and music*, or *mathematics and architecture*, or *mathematics and computers*, many liaisons have emerged from these seemingly unrelated disciplines.

One particular area which has unfortunately been neglected or restricted for reasons unknown to me, is the relationship between *mathematics and sex*. Although similar titles exist, most have focused on the issue of mathematics and gender, or some feminist critique of male chauvinism in mathematics, depriving the weaker sex of equal opportunities in pursuing higher degrees in the subject.

In the next few pages I have attempted to put forward some parallelisms between mathematics and sex, many pertaining more to an Asian milieu rather than an American or European surrounding. But readers should find common ground in many of the ideas formulated regardless of their backgrounds.

MATHEMATICS	SEX
Mathematics is associated with strong imagery (numbers, symbols, formulas...).	Sex is associated with strong imagery (organs, physiques, fantasies...).
Mathematics is more about doing rather than reading and knowing.	Sex is more about doing rather than reading and knowing.
Mathematical knowledge is not of great use unless it arises from, or can be integrated with, experience.	Sexual knowledge is not of much use unless it is accompanied by practice and experience.
Success in proving a theorem is immensely satisfying.	Success in attaining orgasm is immensely satisfying.
Parents take an unhealthy interest in children's progress in mathematics.	Parents take an unhealthy interest in children's sexuality.
Fear of mathematical performance.	Fear of sexual performance.
Math difficulties—help offered by tutors.	Sexual difficulties—help offered by counselors.
Math shortcuts by tutors offer temporary remedies for the learner.	Sexual shortcuts by dealers via drugs and aphrodisiacs provide short-term gratification.
Doing and discussing math with others is more pleasurable than doing it alone. Moreover, having one's assumptions challenged is intellectually satisfying.	Masturbation can be rewarding and enjoyable, but having sex with another human being is more satisfying for most people.

Learning about mathematical techniques equips one to be a better problem-solver.	Learning about sexual techniques increases one's enjoyment and effectiveness.
Specific mathematical disabilities are overcome by behavioral techniques.	Sexual dysfunctions are helped by behavioral techniques.
Use of drugs to enhance the patient's capacity to better memorize mathematical formulas and proofs for reproduction in the examination.	Use of hormones to make up for any deficiency in the body's hormonal level to enhance sex drive for achieving sexual bliss.
We all make use of our mathematical awareness in dealing with the world (and with relationships).	We all make use of our sexuality in our non-intimate relationships.
Bogus mathematical and statistical arguments persuade us to accept political or social decisions.	Advertisements using sexy models persuade us to accept the sexual messages.
Exploitation by the unscrupulous to terrorize the mathematically challenged.	Exploitation by the mass media to appeal to the sexually hungry.
We engage consciously in math to improve our problem-solving skills. However, we can become obsessed by performance goals.	We engage consciously in sex to improve our techniques. However, we can become unhealthily obsessed by performance goals.
Cross-fertilization of ideas provides insights for producing new theorems.	Fertilization of cells produces offspring.
Mathematics develops the mind.	Sex develops the physical muscles.
Use of imageries (multi-dimensions) provides ideas for proof.	Use of sexual (masturbatory) fantasies provides ideas for enjoyment.
Innumeracy leads to deterioration of the mind in the long run.	Abstinence leads to impotence in the long run.
Model-answer books are written to help readers gain confidence in answering exam questions.	Sex-help books are written to help readers gain confidence in enhancing their sexual life.
Solving a problem in the external world resonates within the mind itself and relieves internal tensions.	Attaining orgasm from the sexual act resonates within the mind itself and relieves internal tensions.
Cheating on a math test/exam fulfills a hidden desire to score.	Molestation & accessibility to yellow literature and blue desire movies fulfill a hidden sexual desire.
Words like probability & statistics, permutations & combinations, and the like conjure up images of fear and disgust.	Words like anal sex, oral sex, venereal diseases, and the like conjure up images of fear and disgust.
In male teenagers, experiencing a "wet dream" releases tensions and anxieties of math exams.	In male teenagers, experiencing a "wet dream" discharges sexual tensions and anxieties about having sex.

Nightmares involving sitting for math exams and low scores fulfill the dreamer's fears.

Nightmares involving illicit sexual encounters (forbidden acts) fulfill the dreamer's fears/desires.

Mathematical maturity: the participants solve a mathematical problem in the shortest time.

Sexual maturity: the participants achieve sexual bliss simultaneously in the shortest time.

The study of mathematics provides the best remedy against the lusts of the flesh.

The study of sexology provides the best remedy against the abstractness of mathematics.

Intercourse between different branches of mathematics produces new results which are often beautiful, shorter and useful.

Intercourse between people of different races produces offspring who are often prettier, stronger and smarter.

Mathematics provides an oasis for the sexually frustrated.

Sex provides an oasis for the mathematically frustrated.

A willingness to get away from conventional mathematical thinking may lead one to an unexpected elegant result.

A willingness to get away from conventional sexual practices may lead one to experience an unexpected bliss.

Abstract mathematics is repulsive to immature young minds.

Unorthodox sex knowledge is repulsive to immature young minds.

A sexually mathematical joke characterizes the intercourse between reason and unreason.

A mathematically sexual joke characterizes the tension between sexual fulfilment and deprivation.

Very little of mathematics is useful practically for numerical competence, and that little is comparatively dull.

Very little of sex is useful practically for reproduction, and that little is comparatively dull.

Number worship has diabolically enticed the mathematically ignorant.

Sex worship has diabolically lured the sexually ignorant.

Magic squares provide recreational enjoyment for the mathematical mind.

Magic squares engraved on talismans are used to relieve pain during childbirth.

Mathematics is about trust and understanding: Pure mathematicians don't trust applied mathematicians, and applied mathematicians don't understand pure mathematicians.

Sex is about trust and understanding: Women don't trust men, and men don't understand women.

Pure math (number theory) has prostituted itself into applied math for the sake of fulfilling the practical wants of mankind (e.g., use of prime numbers in cryptography).

Men and women have prostituted themselves for the sake of fulfilling the sexual desires of mankind.

Idiot-savants boast about their extra-numerical skills.

Don Juans boast about their extramarital adventures.

Applied mathematicians are often regarded as second-hand mathematicians as compared to pure mathematicians who are ranked first-class.

Male and female prostitutes are often regarded as "second-hand goods" as compared with those who lead a monogamous lifestyle.

Mathematics may be abused to meet the selfish whims of the military in developing lethal weapons to annihilate the enemy.

Sex may be abused to meet the selfish whims of sex perverts by forcing their victims to perform fetish acts.

Mathematics has come to symbolize all social ills through technological advancement.

Sex has come to symbolize all social ills through the spread of incurable diseases (AIDS, cancer...).

Mathematics is the No. 1 mental killer of all academic disciplines in society.

Sex is the No. 1 emotional and physical killer in society.

Mathematics under untrained teachers has led to a high rate of drop-outs among its practitioners.

Unprotected sex has led to a high rate of diseases among its practitioners.

One must be old enough to understand abstract mathematical concepts, e.g., topology, infinity, and Hilbert-space.

One must be old enough to make sense of abstract sexual behaviors, e.g., sado-masochism, pederasty, and homosexuality.

Math manipulatives (Soma cubes, abacus...) meet the desires of the mathematically curious.

Sex gadgets (vibrators, dolls...) meet the desires of the sexually curious.

Numerology is exploitative mathematics, but abstract mathematical ideas are the quintessence of the mind.

Pornography is exploitative sex, but sexuality is art.

Factoring a quadratic becomes confused with genuine mathematical talent.

Fathering a child becomes confused with genuine sexual virility.

Mathematical results are often sexualized:
$\int e^x = f(u^n)$
"Sex is quadratic." Stanislaw Ulam
Some graphs conjure up images of sexual organs.

Sexual behaviors and positions are often quantized: 6, 7, 40, 69, ...

Mathematics is a dangerous activity; an overdose can result in madness. It undermines one's mental health.

Sex is a dangerous activity; an overdose can result in cancer. It undermines one's physical and mental health.

Mathematics extols rigor.

Sex extols vigor.

Lying in math is common: you never tell the whole truth to simplify your explanation.

Lying about sexual matters is common: you never tell the whole truth to avoid hurting your partner.

The cardinal sin of division by zero results in a mathematical breakdown (fallacy & paradox).

The cardinal sin of adultery results in a marital breakdown (divorce & separation).

Mathematics is predominantly a man's activity.

Sex predominates a large portion of a man's brain.

Going through the details of a mathematical proof is boring and uninteresting, but giving the gist of it is enlightening.

Watching the detailed motions of a sexual act is disgusting and unexciting, but watching an erotic movie is more satisfying.

Men dislike women who are mathematically active.

Men dislike women who are sexually inactive.

There will always be a more beautiful theorem and a more elegant proof.

There will always be a prettier woman or playboy.

The only mathematical activity known to the average person is that of reproducing work for examinations.

The only sexual activity known to the average person is that of providing egg or sperm cells for reproduction.

Mathematical magic relegates mathematics to number mysticism.

Sex shows relegate the participants to mere sexual objects.

Mathematics is used as a powerful tool to destroy the enemy.

Sex is used as an enticement to lure the enemy.

Mathematics (solid foundation) opens the door to many high-paying jobs.

Sexual offers (feminine charm) may open the door to many high-paying jobs.

Cybermath will revolutionize the teaching and learning of mathematics. Virtual mathematics will substitute tutoring.

Cybersex will revolutionize sex education. Virtual sex will substitute prostitution.

The aha-feeling of being able to solve a challenging problem.

The oomph-feeling of both partners attaining orgasm simultaneously.

REFERENCE

Ball, Derek (1993) "Mathematics and the Oedipal Struggle," in *For the Learning of Mathematics* **13**, *1* (February, 1993), pp. 4-5.

Imaginary

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May be sung to the tune of "Imagine" by John Lennon.

Imaginary numbers, multiples of i
Everybody wonders, "are they used in real life?"
Well, try the amplifier I'm using right now—A.C.!

You say it's absurd,
this root of minus one.
but the same things once were heard
About the number negative one!

Imaginary numbers are a bit complex,
But in real mathematics, everything connects:
Geometry, trig and calc all see "i to i." Ah-hai!