The Classroom Encounter

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As every newspaper reader knows, many people are trying hard to reform math education. The NSF, NCTM, MAA, and AMS are helping. Meetings are held. Grants are awarded. Textbooks are written and rewritten. "Technology" (use of calculators and computers) is introduced and expanded.

Has this activity made significant improvement?
"Too soon to tell."
When will be the right time to tell?

WHAT'S MISSING?

I hope this work succeeds, but I'm not optimistic. Why not? Because the reforms concentrate on *curriculum* and *teaching strategy*. The *encounter between teacher and student* is underestimated.

In the AMS *Notices* (1) Hyman Bass recently wrote: "Mathematical scientists typically address educational issues exclusively in terms of subject matter, content and technical skill, with the 'solution' taking the form of new curriculum materials. Curriculum is, indeed, a crucial aspect of the problem and one to which mathematically trained professionals have a great deal of value to offer. But, taken alone, it can and often does ignore issues of cognition and learning."

Of course the classroom is a place where information is transferred, but before that, it's a place where humans encounter each other—student with student, teacher with student. The successful teacher relishes that human encounter. He/she knows that teaching isn't just copying information from one abstract intelligence to another. "Covering the material" doesn't necessarily mean teaching the students. Human feelings and needs affect academic performance. In fact, research and experience have found a key ingredient in successful teaching—the relationship between teacher and students-sometimes called the "affective" aspect.

We don't talk about this very often. It's not stressed in our teaching literature, whether "conservative" or "reform."

Of course lectures should be correct, comprehensible, and interesting. But it also matters whether the student sees the teacher caring about her/him, as a human being.

What does it mean to care about the student? It means of course caring whether the student follows the lectures and does the problems. It also means caring why the student is in the class, and what his/her background, preparation, aspirations are. It even means caring when he/she has a crisis—in health, family, employment, or a significant relationship.

When a student has a crisis, does he/she have a reason to believe the professor would want to help?

More primitive: does the professor seek eye contact with the student, or avoid it? Does he/she talk to the students, or to the blackboard? When he loses the class, does he notice, and reestablish contact, or just go on obliviously? Does he let the students see him as a human being with feelings, needs, and weaknesses, or does he try to impersonate a talking, writing automaton?

Some mathematicians laugh such considerations out of court as "pedagogy." Some cry derisively, "Touchy feely!" I'm afraid that even the word "caring" is considered out of place in a mathematical publication. That very fact is a telling indication of the problem I'm talking about

I'll quote three sources. First, an MAA pamphlet of a quarter century ago. Second, a fantastically successful undergraduate math program in Potsdam, N.Y. Third, a study by two anthropologists on why undergraduates switch out of science, math and engineering.

MAA SUGGESTIONS

When I was young the *Monthly* had a section, "Classroom Notes," which accepted pedagogic contributions. It's no longer there. Maybe it was judged insufficiently mathematical. Paul Halmos wrote (12), regarding educational contributions to the *Monthly* when he was editor: "if the educational wisdom that an author had to offer made sense when the word 'mathematics' in it was replaced by 'geography,' say, throughout, then, I said, it should appear in a journal devoted to education, not mathematics. My aim, to make everything in the *Monthly* mathematical." But the readers of the *Monthly* don't read journals devoted to education. So, they're deprived of the wisdom in question.

I found an MAA publication of 1972 which had good advice for any math teacher: *Suggestions on the Teaching of College Mathematics*, credited to a committee chaired by D. W. Bushaw (2).

> "...the perceptive teacher who looks at students while he talks can hardly miss signs of puzzlement, boredom, or pleasure on their faces...if you sense that you have 'lost' a student you might pause and ask him if something needs further explanation. But a word of caution: impatience on your part with the nature of the student's question may result in an impassive and unreadable face on that student for the rest of the term....Another type of feedback that may be especially useful to the inexperienced teacher is obtained by spending the last few minutes of each period discussing what went wrong and what went right that day. LISTEN CAREFULLY to what the students have to say, even if it seems unreasonable...If the class was dead, say so. Make it clear that the students share responsibility if the class is a drag.
>
> "Every good teacher wants rapport with his class, but it is amazing how many instructors give their lucid explanation to the blackboard, the walls, a window, or a point about one foot over the students. LOOK THEM IN THE EYE!
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> "Encourage conjectures and do not ridicule inept questions or wrong answers. Give the students the feeling that they are all on an equal footing in your esteem. You can learn from their mistakes. Above all, avoid sarcasm in any form. Nothing can damage your relationship with a class faster than sarcasm, however warranted it might seem.
>
> "LISTEN TO YOUR STUDENTS. When someone volunteers an answer to one of your questions, you may realize as he begins to talk that he is on the wrong track. Resist the urge to quiet him. Instead, try to understand what he is saying, acknowledge any merit in it, determine his misconceptions, and tactfully point them out to him. Then let him try again, or give someone else a chance. Many instructors misinterpret a question before it is completely formulated. After you have tried to answer a question, give the student who asked it a chance to say whether he is satisfied.
>
> "In general, strive for as much informality in the classroom as your own personality and the circumstances will allow. Don't be defensive when you make a mistake. No one is perfect, and an impression of integrity is more important than an impression of omniscience. Request help from the students and correct the error together."

In my opinion, asking every math instructor or professor to read this little book once a year would do more for math education than several committee meetings on calculus reform.

There have been many recent publications about college math teaching, mostly from the MAA. In particular, (4), (5), (7), (9-11), (15), (16), (19), (20), (22-29). These and others have excellent suggestions about curriculum and teaching strategy. Most of them don't strongly emphasize the relation between teacher and student.

A recent MAA publication that deals substantially with the teacher's interaction with students is *Keys to Improved Instruction by Teaching Assistants and Part-Time Instructors*, edited by Bettye Anne Case (3). It includes an anthology of guides for TA's and part-time and temporary instructors, collected from 10 universities. This is very good. Of course, TA's and part-timers aren't the only ones whose teaching can improve. But, unlike tenured faculty, TA's and part-timers do have to listen when told to improve their teaching.

Is the teaching of tenured faculty above criticism? An anecdote from a prestigious East Coast school: A student complained that Professor X treated him unfairly in an oral exam. The chairman gave the student a second exam. The student did about as badly as the first time. So far so good. But then the chairman told Professor X what he had done! Professor X was indignant to the point of never again speaking in a civil manner to the chairman. Moreover, X's wife and the chairman's wife had been friends. End of friendship. Probably an extreme case.

"A MODERN FAIRY TALE"

My second evidence is a remarkable article from the *Monthly* March, 1987. It's titled, "A Modern Fairy Tale" (20). It describes an amazingly successful undergraduate mathematics program at a little known college, Potsdam College of the State University of New York.

Potsdam is a small town in far northern New York State. It's the home of Clarkson University, formerly Clarkson Institute of Technology. The author, John Poland, is in the Department of Mathematics and Statistics of Carleton University in Ottawa, Canada. He wrote:

> "Tucked away in a rural corner of North America lies a phenomenally successful undergraduate mathematics program...Picture a typical, publicly funded, Arts and Science undergraduate institute of about 5,000 students, with separate departments of Mathematics and Computer Science. While the total number of undergraduates has remained relatively fixed over the past 15 years, the number of mathematics majors has doubled and doubled again and again to over 400 now in third and fourth year. They don't offer a special curriculum...It is just a standard, traditional pure mathematics department.
>
> "More than half the freshman class elect calculus, because of the reputation of the mathematics department carried back to local high schools. And, of the less than 1000 Bachelor degrees awarded, almost 20% are in mathematics. In case you are unaware, 1% of Bachelor degrees granted in North America are in mathematics. These students graduate with a confidence in their ability that convinces prospective employers to hire them, at I.B.M., General Dynamics, Bell Laboratories and so on...
>
> "Do they just lower their standards? Mathematics teachers in the university across the street say, 'no.' They see no significant difference between their performance and that of their own students....
>
> "The students say the faculty members really care about them, care that each one can develop to the maximum possible level...*It is simply the transforming power of love, love through encouragement, caring and the fostering of a supportive environment*...By the time they enter the senior year, many can read and learn from mathematics texts and articles on their own...They graduate more women in mathematics than men. They redress a lack of confidence many women feel about mathematics. In the past ten years, almost every year the top graduating student at this institution, across all programs, has been a woman in mathematics.
>
> "What must a mathematics department do to attain this success? The faculty must love to teach, with all this means about communication, caring for students and for their development They would teach at a pace which allows students time to struggle with the problems and resolve them, rather than primarily to cover material...They would recognize that students need time to build the skills, understanding and self-confidence to handle more advanced mathematics. The faculty would encourage and reward the successes of the students, bringing all or most of them to a high level of achievement (and high grades), rather than using the grade to filter the brightest and quickest students into further mathematics studies. The recipe for success at Potsdam is very simple: instill self-confidence and a sense of achievement through an open, caring environment."

The atmosphere and attitude at Potsdam are largely the creation of Prof. Clarence Stephens, who was chairman of the math department and gradually remolded it according to his vision. Prof. V. C. Cateforis, the present chairman, says that changes in the incoming freshmen have diminished the number of math majors. But, the teaching philosophy is still the same. In every course, the expectation is still that all students will learn to write correct proofs, a goal some other departments would think hopelessly unrealistic.

When Poland's article appeared in 1987, 1 expected a sensation. Many other departments would seek to emulate Potsdam, I imagined. Alas, no. Even though Potsdam was held up as an example in Leonard Gillman's retiring presidential address (10) to the MAA, less than half a dozen math departments sent visitors or observers. I don't know of one that succeeded in following Potsdam's example.

Why?

TALKING ABOUT LEAVING

My third document is a book, *Talking About Leaving: Why Undergraduates Leave the Sciences*, by Elaine Seymour and Nancy M. Hewitt (23). Seymour directs ethnography and assessment research in the Bureau of Sociological Research, University of Colorado, Boulder.

They asked, why do 40 to 60 per cent of undergraduates leave science, mathematics and engineering (SME) majors? They studied seven four-year institutions of seven different types.

> "We discovered that the same set of problems lead both to switching and to serious discontent among those who persist.
>
> "...What distinguishes the survivors isn't the nature of their problems, but whether they're able to surmount them quickly enough to survive. The concerns of both switchers and nonswitchers are the same issues across all seven campuses, regardless of size, mission, funding, selectivity, or reputation. In contrast to the common assumption that most switching is caused by personal inadequacy in face of academic challenge, we find that a high proportion of switching arises from institutional sources, or from students' career concerns.
>
> "Ranked by their contribution to switching, these causes are:
> • loss of interest in science
> • belief that a non-SME major is more interesting or gives a better education
> • poor teaching by SME faculty
> • feeling overwhelmed by the pace and load of curriculum demands.
>
> "Criticisms of faculty teaching contribute to a third of all switching and were the third most common factor in switching decisions. Complaints about poor teaching were near universal by switchers (90.2 percent) and were the most common complaints by nonswitchers (73.7 percent)... Rejection of SME careers is partly rejection of the models which SME faculty and graduate students present to undergraduates. SME faculty are often seen as unapproachable or unavailable for help with academic or career planning concerns.
>
> "The curve-grading widely used by SME faculty is perceived to reflect disdain for the potential of most underclassmen. This grading is seen as intended to drive most students away, rather than to give students realistic feedback.
>
> "Harsh grading is part of the traditional competitive SME culture. It discourages collaborative learning, which many students view as critical to understanding the material...
>
> "Students [made] inferences from faculty teaching:
> • Faculty find the subject dull.
> • They have little understanding of how people learn.
> • They dislike teaching, don't care about students.
> • They don't see themselves as responsible for students learning.
>
> "Students didn't believe there was anything intrinsically dull about the SME class material. Same material, different professors, different outcomes."

Seymour and Hewitt checked the widespread notion that TA's with poor English drive students away. According to the students, TA's are not the problem.

There are bad teachers in all subjects, but we seem to have more than our share. Why is this? Why is bad teaching so persistent?

A SOURCE OF BAD MATH TEACHING

I'm talking about teachers at universities with PhD. programs in math. Liberal arts colleges seem to be different

For many of us the passage through graduate school was deeply imprinting. We were apprentices, struggling for our thesis adviser/supervisor's approval. This apprenticeship stamped many of us with our adviser's way of thinking and teaching. (Occasionally the imprint was reversed. After a "stormy" advisership, a student sometimes teaches and thinks in a style opposite to her adviser's.)

In research this tendency is well known. The experienced reader recognizes the writing, not only of Professor X, but also of X's students. It's natural that something similar happens in teaching. This is rarely mentioned, because teaching is semi-private. (Not strictly private, since students are present. But to the professor's colleagues, it's private. Mathematician A generally doesn't know much about the teaching of mathematician B.)

Graduate math teaching seeks to produce mathematicians. If some students get Ph.D.'s, publish and become recognized mathematicians, the program is a success. If others fail to follow the lectures or complete the program, that's of little consequence.

A successful graduate professor is embedded in research. In his graduate teaching he may use the language, assumptions, viewpoints he does with research colleagues. Then the graduate student must somehow leap into the gestalt of research level talk.

There's a connection between teaching and writing. A tragic policy of some math research journals is to severely limit motivation and heuristics. Authors are not encouraged to write much about why their problem is interesting. Even less may they describe the blind alleys that ultimately led to success. From a certain "rigorous" point of view, it's necessary only to state theorems accurately and prove them correctly (rigorously.) Where they come from, what they're good for, aren't part of the mathematics. Indeed, the graduate professor himself need not have a deep understanding of where his subject came from, or what it's good for, if he was educated in the abstract, dogmatic style he perpetuates.

"

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His lectures can be as bare of heuristics and motivation as his articles. Consciously or unconsciously, his students can take him as a model. While taking his course they work as teaching assistants. Often they are given no training in teaching or lecturing. They're just handed a textbook, a classroom number and a meeting time. The graduate lectures they attend every day affect how they teach their calculus or pre-calculus students.

Some TA's are naturally good teachers. Some others learn in time to listen to students and communicate with them. This is a personal matter. The typical university neither requires it nor rewards it.

Later, as assistant professors, they are free to continue teaching in the style they started as TA's. After all, nobody says to do different. Their first concern now, of course, is tenure, not teaching. (Their students do evaluate their teaching. But students usually can't explain very well what they don't like. Anyhow, evaluations don't matter much if they aren't catastrophic.)

This description of untenured assistant professors doesn't apply to participants in "Project NExT". This exemplary MAA activity brings them together and helps them exchange ideas and experiences about teaching and other professional concerns.

To be sure, some graduate math professors are great teachers who love to explain the heuristics behind their discoveries. Inspiration from such a professor can persist in teaching by his student, just as the dogmatism of another professor can persist in teaching by his student.

Not all graduate math teachers are inspiring. To join a graduate faculty of math you're not necessarily required to be a great teacher. What you do in class is pretty much your own business. Take pains with your teaching or don't take pains, most of your colleagues will be neither delighted nor upset.

I conclude that major obstacles to reform of math teaching are the teaching styles we absorb in graduate school, and the policies of our institutions that under-value teaching quality in hiring, tenure and promotion.

WHAT CAN BE DONE?

Can we change this story, where bad teaching propagates from one generation to the next?

We could concentrate on the leaders of American math. Top math professors in top grad schools, research managers in top industrial labs, top math bureaucrats in the U. S. Office of Education and the 50 State Departments of Education, editors of math texts in top math text publishing companies, math ed. professors in top Colleges of Education, top officers and staff of AMS, MAA, SIAM.

These men and women might come to agree and declare "It matters *how* math is taught, not just *what* math is taught. Treat math students as human beings, to avoid math avoidance. Independent work is important, K through 20. Realistic, credible applications are important, K through 20."

Such an agreement and declaration would result in improved math teaching.

Can we bring such a solution about? It doesn't seem easy. Recall Aesop's fable. To be safe from Kitty's claws, the mice must hang a bell on her neck. But which mouse will bell the cat?

Maybe we can improve mathematics education by organized effort, by education, by long-continued lobbying and agitation. That's how change is usually achieved in the U.S. Who will do that organizing, educating, lobbying and agitating? Those who care enough. I know of one small organization* in this work today (reference below.) If enough people care, more organizations may appear. More people may join. Something may happen.

But while we try to transform math education in the large, let's change it in the small. Let's teach students (not merely "teach the material"), by knowing them and caring about them (as far as class size permits!) Let's understand where the math came from and where it's going, and share this information with our students. Let's insist on interaction in the classroom, not tolerate passive classes that just copy formulas off the blackboard.

To change an old saying, "Let's light a candle or two while we curse the dark."

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Editor's Note: HMNJ #8 includes three articles about Potsdam College.

PROPOSED RECOMMENDATIONS, FROM THE "MATHEMATICS AND THE MEDIA CONFERENCE", TO EDITORS OF MATHEMATICS JOURNALS AND MAGAZINES.

1. The *Monthly*, the *Notices*, and other publications should have a monthly column on teaching mathematics. The privilege of writing a column would be awarded to teachers in all sorts of institutions who are nominated and selected as outstanding teachers.

The purpose is not only for the value of the columns, but especially as national acknowledgment of teaching as a high-prestige activity. National recognition would foster local recognition of teaching as a high-prestige activity. Local recognition would be an incentive for people to pay attention to their teaching.

2. "Suggestions on the teaching of college mathematics," produced 25 years ago by Don Bushaw and a committee, and published by the MAA, is an outstanding guide book on college math teaching. It's out of print and almost forgotten. It should be reprinted, possibly with some additions, and marketed enthusiastically. The *Monthly*, the *Notices*, etc., should print a reminder every August that a conscientious mathematics teacher ought to reread this booklet at the beginning of the school year.

Letter to *The New York Times*

Arthur B. Powell
Miriam Yevick

Dear Editor,
As respectively retired and current professors of mathematics at Rutgers, the State University of New Jersey, who taught and teach remedial mathematics besides other mathematics courses at the college level, we should like to express our extreme disagreement with the intention to drop such programs from colleges as reported in *The New York Times* ("Pataki-Giuliani Plan Would Curb CUNY College's Remedial Work," May 7, 1998).

University College on the Newark Campus (now administratively absorbed into the regular day school) is a special division devoted to part-time students who take an average of seven years to earn their degree. A large percentage of the students are from racial and ethnic minority groups and over half are women. The college is highly successful in propelling this student population of working adults from low-skilled wage earners into a broad variety of skilled and professional fields. With their better paying jobs, they more than repay national, state, and local governments in the form of higher tax contributions.

Even though I (Professor Yevick) am a research mathematician with a Ph.D. from M.I.T., I found this teaching, in which I engaged for some twenty-five years, highly rewarding. I felt great admiration for women who would get up at 5 a.m. to prepare dinner in advance, get the kids off to school and then head out to full-time jobs, and thence go to school and do homework at night determined to get that degree. Although their mathematics background was abysmal and the students suffered from extreme math anxiety, I was able to inspire in them a sense of self-confidence. My greatest satisfaction was to see a student's face light up with a flash of insight when I conveyed the meaning of mathematical concepts to them such as, for example, the notion of x's and y's in analogy with pronouns. I even developed a special course "Mathematics for Life and Society" (See *Technology Review*, October 1984, p. A21.) in which the basic skills were extracted from application to social and economic problems relevant to the daily lives of my students. Thus I overcame the notion that remedial mathematics was a waste of time.

Teaching in the Newark College of Arts and Sciences, the day college, I (Professor Powell), trained in pure mathematics at the University of Michigan, am acquainted with the extraordinary efforts of students required to do developmental work in mathematics in college. For most students in this group their previous scholastic experiences have already severely diminished their academic self-esteem. More particularly, these previous experiences have all but eroded their sense of themselves as capable of understanding, doing, and much less applying mathematics. Nevertheless, during my 17 years of teaching in the day college, I have witnessed many students, once given the opportunity and coupled with innovative pedagogy, decide to engage mathematics beyond the degree requirements. Among these students, some have gone so far as to minor or major in mathematics or computer science. Furthermore, these students have gained invaluable insights into the complex task of learning mathematics. Some even have participated, while students, in research activities as researchers and have reported their results at professional conferences as well as published articles in academic journals and chapters in books on mathematics education.

Such results are only possible within a college setting. Here students can take college level courses simultaneously with remedial and developmental work and experience remediation as a first step towards the attainable goal of a college degree. We never did feel that our students did not belong in college. We can do no less than maintain and expand such programs at the college level to help salvage a student population whose intellectual potential would otherwise be shamefully neglected.

Miriam L. Yevick,
Retired Associate Professor of Mathematics
Arthur B. Powell,
Associate Professor of Mathematics
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Book Reviews: *Einstein: A Life*, by Denis Brian, and *The Silver Horse-shoe*, by Javad Tarjemanov

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Einstein: A Life. Brian, Denis. John Wiley and Sons, Inc.

The Silver Horse-shoe. Tarjemanov, Javad. Graham Whittaker, trans. Raduga Publishers, Moscow.

The lives of two deep, unorthodox thinkers who challenged the accepted ideas of mathematics and science of their day are described in these biographies. There are some parallels in the lives of Einstein and Nikolai (Kolya) Lobachevski, the subject of the latter of the books, and their work resulted in profound and monumental changes in the culture. They both had an uncanny ability to focus with great tenacity on a single idea or problem for extended periods of time to the exclusion of all, or nearly all, else. They both had an aversion, if not contempt, for authority and bureaucracy. Of course, they are far from unique in that respect. They were both confirmed nonconformists. As the esoteric philosopher, Roger Miller, would say, they flushed to the sound of a different plumber. They are connected through their work; Lobachevski's geometry played an important part in Einstein's work on relativity.

With all that has been written about Einstein's life, is another biography necessary? Perhaps not, but I think this one is highly desirable. I have read at least five, and in my opinion Denis Brian's is the most comprehensive and best written. It is also the most humanistic with the possible exception of Kenji Sugimoto's delightful *Albert Einstein: A Photographic Biography*, which might be its equal in that respect. Although Einstein cannot be separated from his thought (he WAS his thought), Brian, with information previously withheld by Helen Dukas, Einsteins personal secretary and Otto Nathan, the executor of his estate, and his in-depth interviews with Einstein's colleagues, was able to give a more thorough picture of the human side of the beloved genius.

Einstein did not fit the pattern of the "normal" from the beginning. He was born extremely fat with a misshapen head. His parents showed concern of possible mental retardation because of his lateness in speaking (although it apparently was not as late as some have implied), and his early teachers showed the same concern because of his slowness in responding to questions. He was a very quiet and withdrawn child, but showed a violent temper in the presence of his sister. This and his high strung and emotional adolescence belied the pacifism and humane demeanor that was to rule the remainder of his life. From early on, Einstein ignored the mundane and all things that bored him, but embraced wholeheartedly things that challenged and interested him. He also began, at this time, his ability for deep, concentrated focus.

I don't know whether this focus on and tenacity with mental pursuits affected Einstein's relationships with the ladies, which Brian chronicles effectively (that I will leave for you to peruse if that sort of thing interests you), but it certainly was detrimental to family relationships. Einstein was a very charming and compassionate person in public and was very well liked by all people of both genders, but was apparently awkward and remote in his relations with family members. He seems to have treated both of his wives and his two sons rather shabbily. Also, after professing great love and showing interest in the daughter born to him and Mileva before their marriage, he seems to have abandoned her without ever seeing her. It is unclear what happened to her, but it is assumed that she was adopted. He also apparently abandoned his son Eduard, emotionally if not financially, who spent most of his life in a mental institution. He appears to have had a fairly active social life. However, he abhorred small talk and carried with him pen and paper; he obviously had his mind on his scientific endeavors. He was an accomplished violinist, and he clearly enjoyed performing with others. An anecdote describes Einstein, upon completing a piece of music, indicating that now he had it, referring to a scientific problem he had been working on. The implication was that his mind had been on science, not on the music he had been performing. I'm not so sure that is the case. It may be, but I think we all have had the experience of a means to a solution of a problem or a way through a stumbling block in a proof of a theorem coming to mind when our mind was totally on something else. I attribute this to the subconscious.

The public has a way of crediting scientific, as well as other, accomplishments to single individuals. Einstein was well aware of and very appreciative of what he had learned from and the motivation he received from others, particularly Ernst Mach, James Clerk Maxwell, and H.A. Lorentz. Scientific thought and developments are chains that wend their way through the culture. Einstein, of course, provided many of the links, but there were a great many others of importance in what Einstein accomplished. Ernst Mach questioned Newton's belief of the absolute nature of space and time (but then so did Leibniz). It was from Mach that Einstein got the idea that empirical statements were statements about sensations. These, of course, were keys to Einstein's thought. Michael Faraday discovered the relationship between electricity and magnetism. Maxwell then put this in precise mathematical form. This was a giant step, and probably the most important link in the development of modern physics. It freed the scientist from the "scientific method" in which laboratory observations were the basis of theoretical work based on induction, and allowed for the use of deduction based on well thought out postulates where the results were then proved (or disproved), sometimes much later, by observation. This was extremely important in the development of relativity theories and quantum mechanics. H. A. Lorentz did the early mathematical work on special relativity, of which Einstein was quick to acknowledge.

Einstein was admired and revered by the public. He was a "pop" hero. Einstein could not comprehend this. How could they have so much admiration and affection for him when they couldn't understand anything he talked about? The only ones who did not have this awe and affection for him were the Nazis and Nazi sympathizers (well, most of them). There was, of course, much discrimination and ill treatment because of Einstein's Jewish heritage. It is ironic that what the Nazis referred to as "Jewish physics," which they denigrated, could have given them a great advantage in the development of the atom bomb, and could have altered the outcome of the war considerably. Not that Einstein was involved in the development of the bomb, for he wasn't. Of course, it was precipitated by his discovery of the relation between mass and energy (which followed from Lorentz's transformation equations), and he was involved in that he, at the urging of Leo Szilard, wrote to President Roosevelt warning of the danger of the Germans possibly developing a weapon and the importance of the U.S. getting there first. However, key players in the drama of atomic weapon development, Lise Meitner, Otto Frisch, Leo Szilard, and others escaped the Nazi terror. Otto Hahn and Werner Heisenberg remained. Hahn was very much against Germany developing a weapon. Heisenberg, apparently, was more interested and capable in the theoretical aspects and the development of nuclear reactors and didn't believe, at first, that a bomb could be developed. However, he did work with the atomic energy project. There is some evidence that he was against the development of the bomb by the Nazis, and he indicated that they (the physicists) used the military for the benefit of physics, not physics for the benefit of the military, but that is another story.

There is much, much more in this comprehensive biography, e.g. , Einstein's political views, his relations with the developing country of Israel, his social life, and his relations with friends and colleagues in Europe as well as at Princeton. He once said that the years at the patent office were his best years; he was not expected to lay "golden eggs." It seems to me the years at the Institute for Advanced Study at Princeton would have been the best, where if he was expected to lay golden eggs, there was apparently little pressure to do so, and he was free to pursue his own agenda at his own pace, which was slow and deliberate but deep and constant.

This was not the case with Nikolai "Kolya" Lobachevski, who was very much overworked with teaching, administrative, and other work while developing his revolutionary non-Euclidean geometry. This book, written by the Russian Javid Tarjemanov, and translated into English, is a gem. It reads like a novel. You are there with Praskovya Alexandrovna, widow of Collegiate Registrar Lobachevski, as she took her three sons, Kolya, Sasha, and Alexie, on their four day trip by horse drawn carriage from their home in Makaryev to Kazan for the purpose of attempting to get the boys enrolled in the grammar school in Kazan. The descriptive narrative is beautifully done, and the illustrations that appear occasionally throughout the book are superb. There was concern whether the boys would be accepted into the grammar school at Kazan, the only one between the Volga and Siberia, a school attended almost exclusively by children of the aristocracy, which the Lobachevski boys were not. They were accepted. However, Kolya, even though the middle brother, was placed, much to his chagrin, in the lowest class while his younger brother was placed with the oldest in the middle class. Kolya, being the smallest, was assumed to be younger than the others and told to sit down and draw pictures while the others were tested. By the time he was tested, he was understandably upset and was treated as a child much younger than he was. I can relate to his feelings. I had a similar experience. It is amazing how supposedly intelligent educators can mistake small size and physical immaturity for intellectual immaturity. Fortunately, I was not in an autocratic system and could ignore this kind of foolish recommendation.

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It is amazing how supposedly intelligent educators can mistake small size and physical immaturity for intellectual immaturity.

Kolya's displeasure at his being put in the low class, his ill treatment by the staff, including being put into a detention cell for something he didn't do, and his general unhappiness, motivated him to escape from the school (using a well thought out plan) and run away. He arrived at the home of the family friends in Kazan just as his mother was packing to return to their home in Makaryev. She honored his request and took him home with her. His mother tutored him in various subjects, and his grandfather, with whom they lived, had a vast library of books that Kolya aggressively perused. After a period of time, a letter from his brother, Sasha, telling how well he and Alexie were getting along spurred Kolya to ask his mother if he could return to school. She agreed, and soon they were back on their four day trek to Kazan. After much pleading by Kolya's mother, the administration allowed him to return to school with the proviso that it was to be at her expense, no more state support. Although there was still much that was unpleasant at the school, the twelve year old Kolya had a much better experience this time around due, to a great extent, to his determination to do so. He had promised his mother not to disappoint her again. He didn't. He spent his student years and much of the remainder of his life at the institution as a teacher and administrator. Well, I am, getting carried away. There is much that is interesting about Lobachevsky's school experience on his non-royal and unconventional road to geometry.

Early on, Kolya's mathematical ability was recognized by many, and early on he was troubled by Euclid's fifth postulate, a trouble that would consume him for a large part of his life. Even though others had tried to reconcile this, and he was advised by others, including Martin Bartels, a former teacher of Gauss who had come to Kazan to teach, not to pursue this; he did.

Did Nikolai have any love in his life besides geometry? Yes, one Anna Yakovian, daughter of one of the University administrators. Her parents were very much against this because he was not of the aristocracy and forbid her from seeing him. She had astutely stated that however much he loved her, science would always be the first lady of his heart. Her father contrived to have Kolya removed from the institution, but fortunately he failed in his attempt. This book ends in 1826 with his presentation to the faculty of his paper, "A Succint Exposition of the Principles of Geometry, with a Rigorous Demonstration of the Theory of Parallel Lines." Consequently, there is no information of a love life between this time and his death in 1856. However, in *Men of Mathematics* E.T. Bell indicated that Lobachevski's health was deteriorating with the death of a son, so apparently there was a wife and family after the years spent when geometry held him in its grip.

The presentation of the paper mentioned above was not well received by his colleagues at Kazan. As Einstein has said, "Great spirits have always encountered violent opposition from mediocre minds. The mediocre mind is incapable of understanding the man who refuses to bow to conventional prejudices and chooses instead to express his opinions courageously and honestly." Probably nowhere is this manifested more clearly than in the reaction of Lobachevski's colleagues to his paper. A dejected Lobachevski felt that they understood nothing about what he said and wondered whether all his effort had been wasted.

As mentioned, the book ends at this point, with the rejection of Lobachevski's work. However, there is an epilogue quoting the message on the coat-of-arms granted to him on 29 April 1838 when he was admitted to the peerage for his outstanding services to science.

One wonders whether geniuses such as Einstein and Lobachevski, with their deep concentration for long periods of time on mathematical and scientific ideas to the exclusion of almost everything else, miss out on a lot of the good things in life. Well, don't we all, especially those of us with mediocre minds? What could be a greater thing in life than being responsible for creations such as theirs?

One also wonders how much of this interesting book, with its great detail of events in the early and middle life of Nikolai Lobachevski, is fact, and how much is fiction. As the artist formerly known as "Fats" often said, "One never knows, do one?" one probably doesn't care too much either. It is a delightful book to read; I'm sure the essentials are essentially true, and it provides a taste of the academic and political environment under which Lobachevski lived, learned, and created.

What of the silver horse-shoe in the book's title? That's your assignment for tomorrow.

"Numbers Man"

Lawrence Mark Lesser

My father got to know my mother while tutoring her in college calculus; this poem is a "math love song" I imagine he could have written for her.

My mother fed me formula, it prob'ly was a sine
I'd grow up to adore ya, your figure and your mind.
So let's go to dinner; I'll compute the tip.
Then we'll go shopping and find the bargains quick, 'cause

One thing to count on, honey, understand
I can handle your figure; lemme be your numbers man!

Give me 4 crayons and I'll color in the map.
We'll find the fast way to Vegas and win big at blackjack,
It takes 7 shuffles to mix up the pack--
Ask me how I know and I tell you that...

One thing to count on, honey, understand
I can handle your figure; lemme be your numbers man!

Hey, I see you hesitating--do I come across as calculating?
But I can take your heart so high--I know so many ways to multiply!

I love equality, and I know the value of place.
I got love that's unbounded 'round this finite space!
Lemme tie your hair with ribbons that only have one side.
Maybe that way your bad hair day becomes a day that's prime!

I'll never say "take a number"
Lemme be your numbers man!

Spirograph® Math

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1. INTRODUCTION

The Spirograph® has been a popular toy for many decades. A search on the world wide web will result in many sites that discuss the mathematics of a spirograph pattern, and your local toy store most likely has an assortment of spirographs, each with its own special feature. The basic spirograph set includes numerous circular discs and annuli. To construct a spirograph design, a pencil or pen is placed in one of a number of perforations on a disc, and a pattern is traced as the disc rolls around a fixed annulus. There are seemingly endless possibilities of designs to be made, including the designs in Figure 1. The construction and comparison of these designs is a rich source of mathematics, adaptable to many levels of students. In this paper we will explore what determines the pattern, and what patterns are possible.

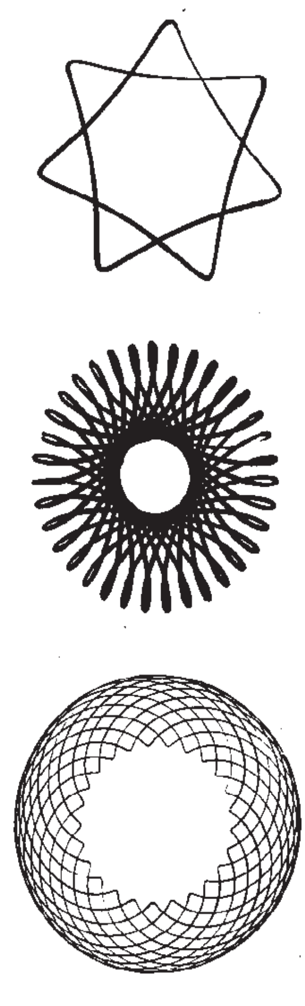

Figure 1
Examples of Spirograph Patterns

The designs made with a spirograph set have mathematical names. In approximately 1600 Gallileo Gallilei gave the name **cycloid** to the curve of the trace of a point on the circumference of a circle that rolls along a straight line. If the tracing point is in the interior of the circle, the resulting curve is called a **trochoid**. In comparison with the cycloid, the trochoid is more flat, as illustrated in Figure 2. In fact, when the tracing point is in the exact center of the disc, the pattern would be a straight line.

A spirograph is a generalization of Gallileo's cycloid. Instead of the circle rolling along a straight line, it rolls along another circle. The curve that is made from rolling a circle inside a fixed circle is called a **hypocycloid** when the tracing point is on the boundary of the circle, and a **hypotrochoid** when the point is on the interior of the circle. If the circle rolls along the outside of a fixed circle, it is called an **epicycloid** or **epitrochoid**.

2. PARAMETRIC EQUATIONS

As well as having mathematical names, these curves also can be described by parametric equations. Most calculus texts include the derivation of these equations in the exercises of the parametric equations section. Denote the radius of the fixed circle by a, and the radius of the rolling circle by b. The parametric equations are determined by the coordinates of the tracing point.

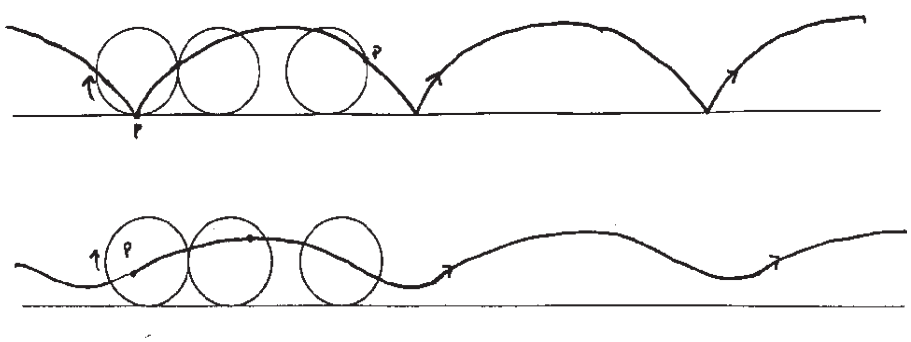

Figure 2
Cycloid and Trochoid

Hypocycloid:

$$x := (a-b)\cos(t) + b\cos\left(\frac{(a-b)t}{b}\right)$$

$$y := (a-b)\sin(t) - b\sin\left(\frac{(a-b)t}{b}\right)$$

The only difference between the hypocycloid and hypotrochoid is the position on the rolling circle of the tracing point. Thus the only change in the formula for the hypotrochoid is the multiplication factor on the second term. The factor is *h*, the distance from the center of the rolling circle to the tracing point.

Hypotrochoid:

$$x := (a-b)\cos(t) + h\cos\left(\frac{(a-b)t}{b}\right)$$

$$y := (a-b)\sin(t) - h\sin\left(\frac{(a-b)t}{b}\right)$$

Equations for the epicycloid and epitrochoid are similar to those above. The main difference is the multiplication factor on the first term. For the hypotrochoid the factor is the difference of radii, *(a-b)*, and this is the distance from the origin to the center of the rolling circle. Since the rolling circle is placed outside the fixed circle for the epitrochoid, the distance from the origin to the center of the rolling circle is the sum of their radii, (a+b).

Epicycloid:

$$x := (a+b)\cos(t) - b\cos\left(\frac{(a+b)t}{b}\right)$$

$$y := (a+b)\sin(t) - b\sin\left(\frac{(a+b)t}{b}\right)$$

Epitrochoid:

$$x := (a+b)\cos(t) - h\cos\left(\frac{(a+b)t}{b}\right)$$

$$y := (a+b)\sin(t) - h\sin\left(\frac{(a+b)t}{b}\right)$$

With the above equations one can use either a graphing calculator or computer software such as Maple to

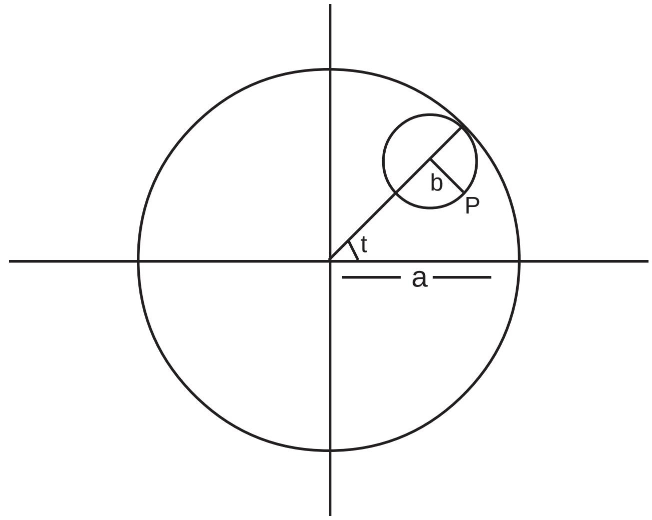

Figure 3
Construction of a Hypocycloid

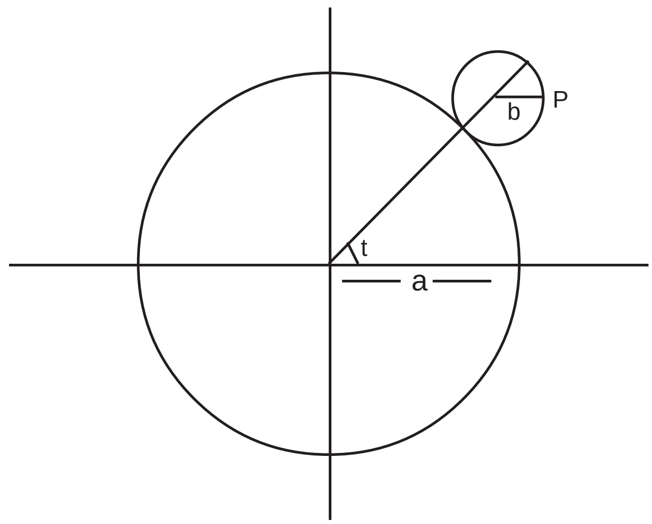

Figure 4
Construction of an Epicycloid

draw spirograph designs without a spirograph set. One can also check the validity of the parametric equations by comparing the computer version with the spirograph version of a particular design. Technically, every spirograph is a hypotrochoid or epitrochoid, since the perforation farthest from the center on any disc is in the interior of the disc.

3. EFFECT OF *a*, *b*, AND *h*

From the equations, one sees that three measurements affect the pattern: the radius *a* of the fixed circle, the radius *b* of the disc, and the offset *h*. First investigate the effect of *h* on the pattern.

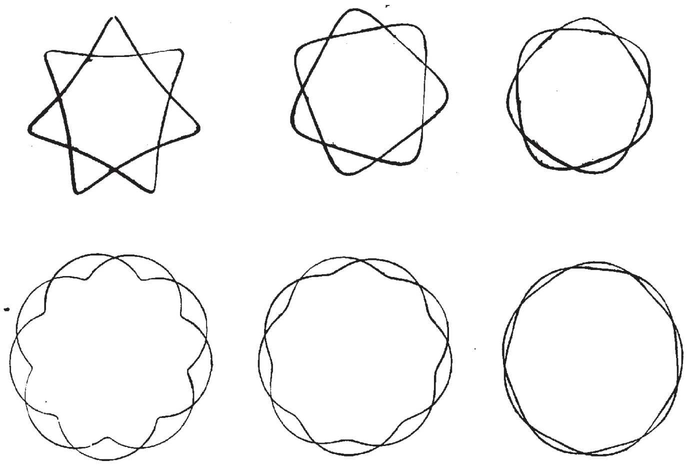

Figure 5
105/30 with offset perforations 1, 6, 8
144/32 with offset perforations 1, 6, 13

The patterns in Figure 5 are indexed with the perforation number on the actual spirograph disc. All the perforations are numbered, with #1 being closest to the circumference and the bigger numbers increasingly closer to the center of the disc. If one chooses a perforation close to the center of the disc, the figure looks more like a circle. If a perforation is at the exact center of the disc, then the offset h is zero, and the equations result in a circle.

In the derivation of the formulas, the radii of the circles were the crucial measurements. In a spirograph set, each circle is numbered according to the number of teeth it has on its circumference. As it turns out, there are 30 teeth for each millimeter of the radius. Notice that the ratio of radii will be the same as the ratio of number of teeth. The figures are labeled according to the number of teeth.

After drawing several patterns, one might want to predict what the resulting designs will be. This is easy using the reduced fraction of number of teeth. In Figure 6 look for a relationship among the designs and the numerator and denominator.

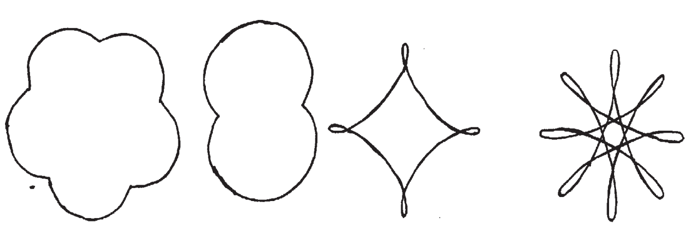

Figure 6
150/30 = 5/1; 144/71 = 2/1; 96/72 = 4/3; 96/60 = 8/5

As one can see, the reduced numerator is the number of cusps. A cusp is made each time the rolling circle completes a revolution. To compute the number of cusps, imagine cutting each circle and laying the resulting straight lines along side each other, as in Figure 7. Paste in copies of each line until their ends match. The number of revolutions that the rolling circle makes is the number of copies of the rolling circle. The resulting equation $s*a=r*b$, describes the situation. Since r and s are the smallest number of copies needed, r/s is the smallest reduced fraction of a/b. Therefore r is indeed the reduced numerator.

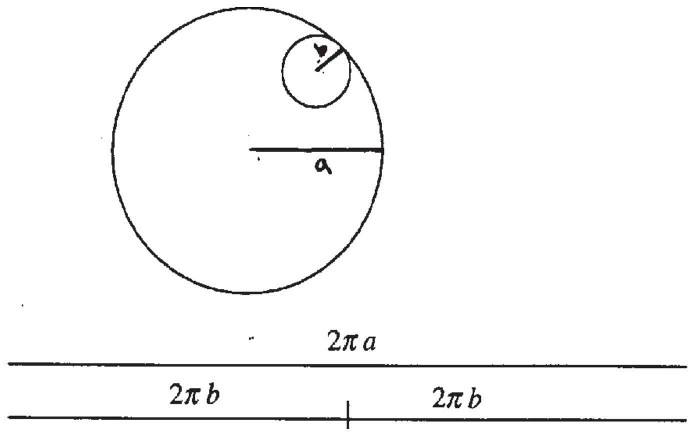

Figure 7
Cut and Flattened Circles

In the above equation s is the reduced denominator and is the number of revolutions of the fixed circle needed for the trace to meet its starting point. This number is called the **period** of the design.

By using the reduced fractions one can determine what kind of figures are possible. In addition to the period and number of cusps, one might recognize a difference in the behavior near a cusp. In some designs the pattern crosses itself before a cusp to form a **node**, and in other designs there is no such crossing. For hypocycloids, when $2b>a$, there is a node for each cusp. For the hypotrochoids, the offset h affects the nodes.

4. SYMMETRY GROUPS

One can also ask what kind of symmetry these designs have. The symmetries of any two-dimensional design are either rotations or reflections, and they form a group with the operation composition. If the designs have both rotational and reflectional symmetry, then the symmetries form a **dihedral** group. A design with

n rotations and n reflections has symmetry group D_n. If the designs have only rotational symmetry, then the symmetries form a **cyclic** group. A design with n rotations has symmetry group C_n. In Figure 8, the seven axes of reflection are included.

Notice that all of the designs thus far have both reflectional and rotational symmetry, so their symmetry groups are dihedral groups. The numerators of the reduced teeth ratios determine which groups are possible. There can be many different designs that represent the same group, illustrating the difference between equal and isomorphic groups. That is, groups can have the same structure but be represented differently. All of the designs in Figure 9 have symmetry group D_5.

What kind of groups occur in a combination of designs? The greatest common denominator of the two numerators give the subscript of the dihedral group. If t is the greatest common denominator of m and n, the number of cusps in each design, then the combination of designs can be divided into t equal pieces. The t pieces will consist of m/t cusps from the first design and n/t cusps from the second design. Thus the symmetry group will be D_t. In Figure 10, the symmetry groups are D_{12}, D_6 and D_5 respectively.

If the reduced fraction of teeth is a whole number, then another kind of design is made by tracing one loop for each perforation in the rolling disc. Such designs have cyclic symmetry groups (see Figure 11). The table of reduced fractions gives that the possible cyclic groups made in this way are C_2, C_3, C_4, C_5 and C_6.

It is also possible to get different symmetry groups from one design by coloring segments formed by cusps. If a design has symmetry group D_n, then there is a coloration that will result in D_m for each m that divides n, and a coloration that will result in C_m for each m that divides n, except $n/2$ and n.

To demonstrate this, take a design of dihedral group D_n. To get D_m, choose 2 colors. With the first color, fill in m equidistant segments. Color the rest of the segments with the second color. D_m can be generated by a rotation of $(360/m)°$ and a reflection across the diametrical axis through one of the cusps with the first color. To get C_m, choose r colors where $n = rm$ and color m consecutive segments with each of the colors,

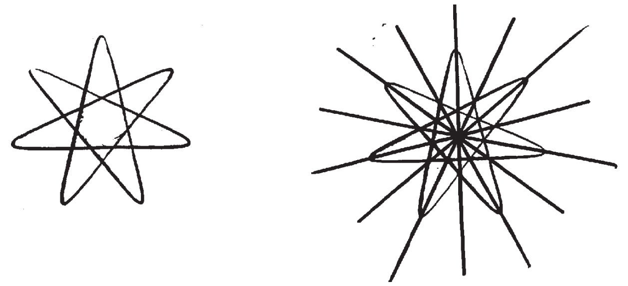

Figure 8
Axes of Reflection on 105/45 = 7/3

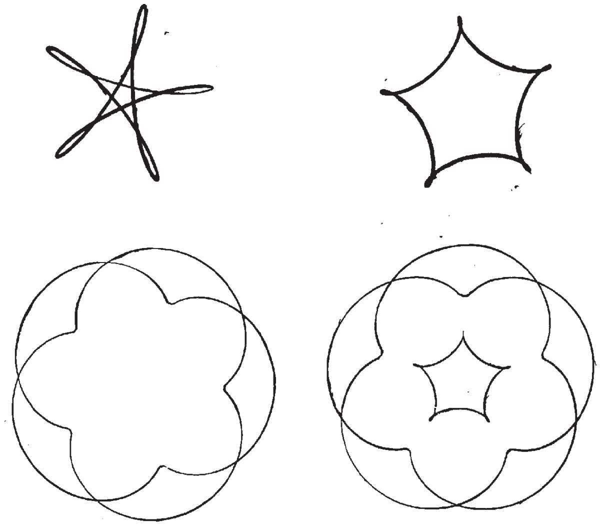

Figure 9
Designs with Symmetry Group D_5
105/63 = 5/3; 105/84 = 5/4; 150/60 = 5/2

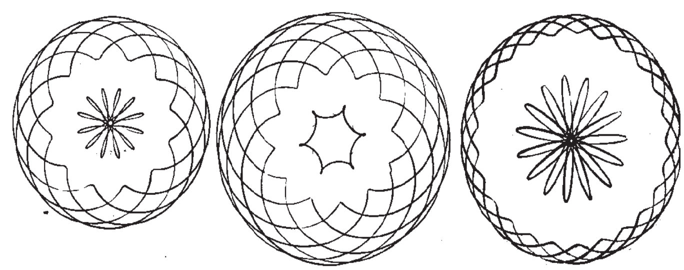

Figure 10
144/60, 96/56; 144/84, 96/80; 150/32, 105/48

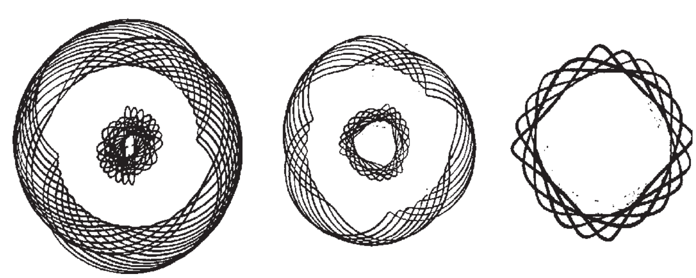

Figure 11
C_2, C_3 and C_4
96/48, 144/72; 96/32, 105/45; 95/24

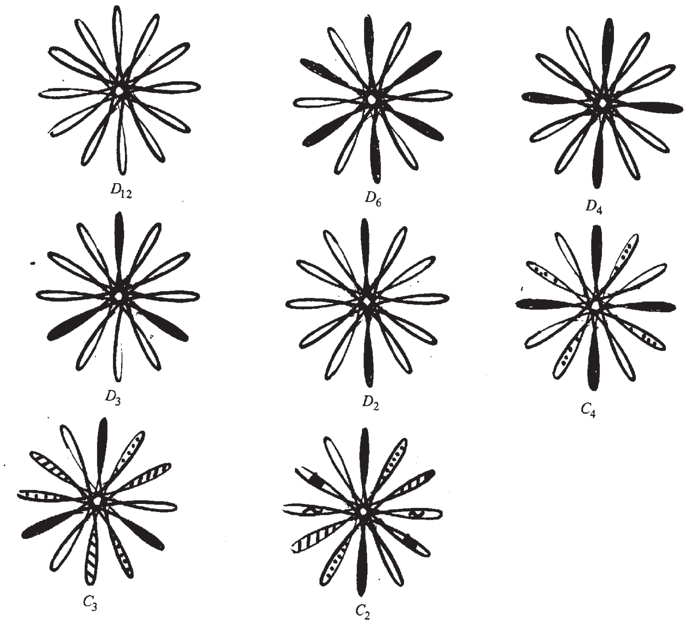

Figure 12
Colorations of 96/56 = 12/7

repeat in the same order until the whole design is colored. This coloration eliminates the reflections, but the m rotations of $(360/m)°$ remain. Notice that when $m = n/2$ or $m = n$, this pattern results in D_m instead of C_m.

While this type of coloring exercise is helpful to the group theory student, it is also meaningful at a more elementary level for a student studying symmetry.

There are other ways to generalize the hypotrochoid and epitrochoid. A spirograph set also contains a rounded square that can be used in place of an annulus. Figure 13 contains two designs made with a fixed rounded square and a circular rolling disc. For such designs, one can determine the kinds of possible patterns, the parametric equations, relationships between the reduced fraction of teeth and the number of cusps and period, and the possible symmetry groups.

Using a previous argument, the number of cusps will be the numerator and the period will be the denominator of the reduced fraction of teeth. Drawing a few designs demonstrates that the symmetry groups will not be D_n, where n is the number of cusps. The designs are patterned after a square, and the largest symmetry group possible is D_4. If the number of cusps is divisible by 4, the design can be divided into 4 equal parts and the symmetry group will be D_4. Likewise, if the number of cusps is divisible by 2 (but not 4), then the symmetry group will be D_2. If the number of cusps is odd, then the symmetry group is D_1, which consists of the identity and one reflection.

Another generalization is to let a and b be any positive real numbers. We have only looked at the designs one can make from a spirograph set, but these formulas work regardless of what kind of numbers a and b are. If they represent irrational numbers, then there won't be a period. Other types of designs occur when b is greater than a. One can get a feel for what designs are possible by going to one of the many Spirograph web sites. One such site can be found at the address:

http://www.wordsmith.org/-anu/java/spirograph.html#def

5. PROJECT QUESTIONS

1. Determine the polar equations for the hypotrochoid and epitrochoid.
2. Determine the parametric equations for a circle rolling inside and outside of a rounded square.
3. Determine conditions of a, b,and h so that singularities occur on a hypotrochoid.

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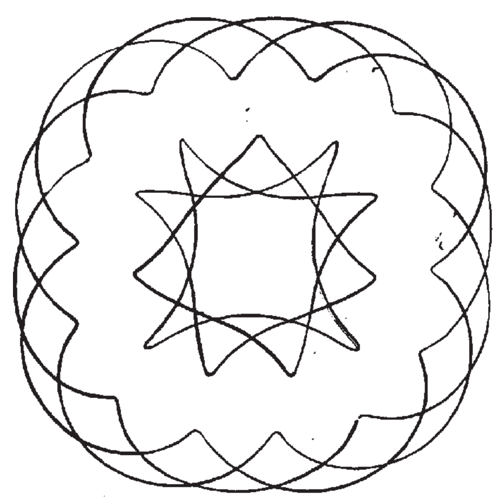

Figure 13
Design Using a Fixed Rounded Square
176/48 = 11/3, 224/48 = 14/3

The Difference Between Pure and Applied Mathematics

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There is none; that's my conclusion. But of course, applied mathematics is mathematics done with some other end in view, whereas pure mathematics is an end in itself. Whether we call some mathematics "applied" depends on what the ultimate end is. For instance, one might say that the theory of solvable groups is "applied" because it can be used to develop Galois theory. This probably isn't what most people have in mind when they talk about applications of mathematics. So only some ultimate ends make mathematics "applied." As an aside, I want to mention that what Morris Kline, in his immensely influential essay[1], refers to as "applied mathematics" could be labelled more accurately as pure mathematics whose subject is nature. He argues that such is the only ultimately worthwhile direction for research, but the question of "utility" is not important for him, so the point I'm making is only tangentially related to his work.

We could say that mathematics is applied if its point is to meet basic human needs such as food, shelter, and warmth. Of course there are derivative needs, which would make some mathematics applied even if it doesn't directly help to supply food, shelter or warmth. For instance, one could argue that military strength is necessary to insure basic human needs, so that mathematics which increased military strength would be applied. To define applied mathematics from here it is only necessary to iterate. Applied mathematics is mathematics that solves problems which (possibly in a highly derivative way) meet basic human needs. For instance, theorems on the plane crossing numbers of graphs have applications to VLSI design, which leads to new computer architectures. Computers expedite food production, and so calculating plane crossing numbers of graphs is applied mathematics.

We can make the same sort of distinction in the field of metal working: Making spoons, thimbles, or counterfeit coins is applied metal work; spoons help people eat and thimbles help them keep warm (in a derivative way). On the other hand, making statues of horses out of bronze is pure metal work. Now we can see the first problem with the pure/applied distinction. People don't just make spoons, they make pretty spoons. In fact, people who make spoons spend a lot of time thinking about making their spoons prettier. This instinct has led spoon designers to make spoons which aren't even meant to be used—they're meant to be displayed. Some people put them in frames and hang them on their walls, so is spoon manufacture pure or applied? The same thing happens with thimbles, and the same thing happens with mathematics.

For instance, as mentioned above, computer scientists study crossing numbers of graphs for the purpose of improving VLSI techniques. Recent work in this area is surveyed in[2], whose authors are all computer scientists, employed in computer science departments, and publishing papers in computer science journals. And yet, they've put a lot of thought into the crossing numbers of graphs on nonorientable surfaces, a study which they certainly aren't pursuing for the sake of improving chip design. My claim is that even mathematics which is begun with the purpose of meeting basic needs will end up as mathematics done for its own sake. The natural end of applied mathematics is pure mathematics, because people naturally want to make even their utilitarian creations beautiful, and the beauty ends up becoming the purpose. The reason for this is that people have a basic need for beauty, truth, spirituality, and a productive life as well as for food, warmth, and shelter. "Man does not live by bread alone, but by every word that proceedeth out of the mouth of God."[3] or "Hearts can starve as well as bodies—give us bread, but give us roses."[4]

Now I want to turn the above line of reasoning upside down. It isn't really true that people begin their applied mathematical activities by trying to meet bodily needs, and subsequently develop them to meet spiritual needs. Actually bodily needs and spiritual needs are inextricably linked, and neither is prior to the other. In primitive societies, where people make their living by hunting and gathering, these are not merely utilitarian pursuits, but spiritual activities—expressions of humanity's place in the universe. If human needs in the large sense are being met, this will always be the case. People won't work effectively merely to feed themselves—If people are left to have their own way, work will always be at the same time an expression of human spirituality. Also, it probably isn't possible to subsist bodily if one's only goal is to get food, warmth, and shelter. Just as a tennis player has to follow through in order to hit the ball effectively, people have to embed their pursuit of bodily requirements in an infinitely richer context in order to be able to meet them at all. Without the larger spiritual context, there's really no point in meeting bodily needs, and without meeting the bodily needs, there's no possibility of the larger spiritual context, so it doesn't make sense to say that one is prior to the other. So in the end there is no difference between pure mathematics and applied mathematics. Both are activities pursued for their own sake, or rather for the sake of living a fully human life.

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Music of the Spheres

Lawrence Mark Lesser

Today was a rainbow day
Horizon leaned the right way
Tonight in moonlight
We sleep beneath the stars we sight
Ohhhh, it's amazing
By the sea, star-gazing
My guiding inspiration all through the years
Has been music and the music of the spheres.

Tides are getting strong
I get my guitar to sing you a song
The notes that bring peace
Are numbers from Ancient Greece
Ohhhh, hear a pattern
In the path of Saturn
I marvel at the harmonies that caress our ears
Sweet music and the music of the spheres

I end my sinple tune
While stars twinkle 'round the moon
Like vibrating strings
We resonate with all these things
Ohhhh, chords and notes,
Words and hopes,
Spinning in the myst'ry of why we cry our tears
For music and the music of the spheres

The Prospects for Mathematics in a Multi-Media Civilization

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(A somewhat expanded version of a talk given at the Urania Theater, Berlin, August 21, 1998, in conjunction with the 1998 International Congress of Mathematicians (ICM 98))

I

I am honored and delighted to be here this evening in conjunction with ICM 98 as part of this most distinguished series at the Urania Theater. I wish to thank the Invitation Committee of the ICM for proposing me as a speaker.

At the very beginning, let me explain my use of the phrase "multi-media" civilization. I mean it in two senses. In my first usage it is simply a synonym for the year 1998 and for the next decade or so. In the my second usage I refer to the widespread and increasing employment of computers, fax, e-mail, CD-roms, chips, videos in all mixtures. I mean it to designate the world that embraces such terms as interface design, cybercash, cyberlaw, virtual-reality assisted learning and V-R-A-surgery, cyberfeminism, tele-immersion, interactive literature, interactive cinema and animation, hand-held electronic books, 3D conferencing, spam.

And I might even include as media the U-Bahn (the subway system), the automobile and modern air transportation that have transmitted all our brains here to Berlin.

I personally cannot do without my word processor and my mathematical software. I find I can check conjectures quickly and find phenomena accidentally. (I find too many trivialities!) I am not yet fully into the Web.

Yes, the computer has become a universal and no longer strange attractor from which none of us are able to escape. Soon we will not be able to read anything without a mouse in our hands. We have been seduced, we have benefited, and we have become addicts.

And what aspects of mathematics shall I consider? The logical chains from abstract hypotheses to conclusions? Other means of arriving at mathematical conclusions and suggesting actions? The semiotics of mathematics? Its applications (even to multi-media itself!)? The psychology of math creation? The manner in which math is done, linked with itself and with other disciplines, published, transmitted, disseminated, taught, supported financially, applied? What will the job market be for its young practitioners? What will be the public's understanding of mathematics? Ideally, I should like to consider all of these. But, of course, every topic that I've mentioned would deserve a week or more of special conferences and would result in a large book.

Poincaré's Predictions

Our proximity to the millennium inevitably suggests that a speaker project forward in time. While such projections, made in the past, have proved notoriously inaccurate, I would be neglecting my duty if I do not make projections even though it is guaranteed that they will become the objects of future humorous remarks.

An example from the past. Ninety years ago, at the Fourth International Congress of Mathematicians held at Rome in 1908, Henri Poincaré undertook such a task. In a talk entitled "The Future of Mathematics," Poincaré mentioned ten general areas of research and some specific problems within them, which he hoped the future would resolve. What strikes me now in reading his article is not the degree to which these areas have been so developed—they have—but the necessary omission of a multiplicity of areas which we now take for granted and which were then only *in utero* or not even conceived.

Though the historian can always find in the past the seeds of the present, particularly in the thoughts of a mathematician as great as Poincaré, I might mention as omissions from Poincaré's prescriptive vision, the intensification of the abstracting, generalizing and structural tendencies, the developments in logic and set theory, the pattern-theoretic, the emerging of new mathematics attendant upon the physics of fluids, materials, relativity, quantum theory, communication theory. And of course, the computer, in both its practical and theoretical aspects; the computer which has altered our lives almost as much as the "infernal" combustion engine and which may ultimately surpass it in influence.

Poincaré's omission of all problems relating immediately to the exterior world—with the sole exception (!) of Hill's theory of lunar motion—is also striking.

> ***The object of computation is not numbers but insight.***
>
> ***--Richard Hamming***

How then can the predictor with a clouded vision and limited experience in such matters proceed? Usually by extrapolating forward linearly current tendencies that are obvious to the most imperceptive observer.

What will pull mathematics into the future?

Mathematics grows from external pressures and from pressures internal to itself. I think the balance will definitely shift away from the internal and that there will be an increased emphasis on applications. Mathematicians require support; why should society support their activity? For the sake of pure art or knowledge? Alas, we are not classic Greeks or 18th century aristocrats, and even their material was pulled along by astronomy and astrology and geography. Society will now support mathematics generously only if it promises bottom-line benefits.

Now focus on the word "benefits." What is a benefit? Richard Hamming of the Old Bell Telephone Laboratories said in a famous epigraph to his book on scientific computation, "The object of computation is not numbers but insight."

Insight into a variety of physical and social processes, of course.

But I perceive (forty years later and with a somewhat cynical eye) that the real object of computation is neither numbers nor insight, but to make money—often via computations that are authorized by project managers who have little technical knowledge. If, by chance, humanity benefits, then so much the better, everybody is happy. And if humanity suffers, the neo-Luddites will cry out and form chat groups on the Web or their hackers will attack computer systems or humans. The techno-utopians will explain that you can't make omelets without breaking a few eggs.

And pure mathematics will follow along, moving closer to applications while justifying its purity to the administrators, politicians and the public with considerable truth that one never knows in advance what products of pure imagination can be turned to society's benefit. Employing that most weasel of rhetorical expressions: "in principle," in principle, all mathematics is potentially useful.

I could use all my time this evening in describing a few applications that seem now to be hot and are growing hotter. I will mention a few and comment very briefly on some of them.

Mathematics and the physical and engineering sciences

Classical. These have been around since Galileo, but only in the past, say, hundred years, has theoretical mathematics been of any great use to technology.

Mathematics and the life sciences

Mathematical biology and medicine are booming. There is automatic diagnosis. There are many models around; most are untested. One of my old PhD students has worked in biomolecular mathematics and designer drugs. He and numerous others are now attempting to model strokes via differential equations of fractional order. Good luck!

Mathematics and the military sciences

Then there is mathematics and the death sciences: war, both defensive and offensive. For the past sixty years this has been a tremendous engine pulling both pure and applied to new achievements. And, of course, defense will be with us as long as aggression is a staple of human behavior.

Mathematics and entertainment

There is mathematics and entertainment, through animation, simulation and computer graphics. The (ex)-executive of Silicon Graphics recently opined that the future of the United States lay not lie in manufacturing nor in the production of food, but in producing a steady flow of entertainment for the rest of the world. Imagine it, a future President of the United States may have to warn us against the media-entertainment complex as Eisenhower did with the military-industrial complex.

But wait! Through animation and simulation, the world of defense joins up with the world of entertainment and the world of medical technology. These worlds find common problems and can share computer software. (There was a recent conference on this topic.) Mickey Mouse flies the Stealth Bomber and performs virtual surgery via the same sort of software products. Young mathematicians take your results not to the ICM but to Steven Spielberg!

Mathematics and money

Marriages of business and mathematics are booming. Business and trade have always been tremendous consumers of low-level mathematics. But now it is no longer low-level. Zebra stripes. (Product identification.) Playing the market by clever statistical strategies. Portfolio management. The United States government spends billions on economic and social modeling and projections.

Mathematics and the graphic arts

Graphic art is being revolutionized along mathematical lines, a tendency—would you believe it?—that was present 3000 years ago in the art of Egypt when some of their art was pixelized.

Computer art shows are now commonplace. Is such art a kind of "soft mathematics?"

Mathematics, law, legislation and politics

Law is just beginning to feel the impact of mathematizations. Leibniz and Christian Wolff talked about this three centuries ago. Nicholas Bernoulli talked about it in 1709. Read his inaugural dissertation "On the Use of Probability (Artis conjectandi) in Law."

Statistics are more and more entering the courts as evidence. There are DNA identifications. Most recent and most notorious identification is the long conjectured Thomas Jefferson—Sally Hemmings liaison. This shows how little one can trust historians who wear blinders fashioned by their preconceptions. But can statisticians be trusted? The "experts" are often found testifying on both sides of a question.

There is epidemiology. There are class action and discrimination suits. Automated multiple regression makes it all feasible.

We spend hundreds of millions on polls of voters, consumers. The census. How to count? The simplest of mathematics operations turns out to be a practical impossibility. Sampling is recommended, but reduces variance, but increases the discrepancy.

On Jan 13, 1998, a demographer, head of the Census Bureau, resigned. She was in favor of sampling. The thought is that sampling will increase the power of the minority party, hence the majority party is against it. On Nov. 30, 1998, the case was argued before the U.S. Supreme Court.

Despite all these developments, we are as far as ever, perhaps further than ever, from Leibniz' dream of settling human disputes by computation.

All of the above major areas are intersected by

Mathematics and education

Consider education, for example. One of my colleagues writes me as follows:

> "My teaching has already changed a great deal. Assignments, etc. go on the web page. Students use e-mail to ask questions which I then bring up in class. They find information for their papers out there on the web. We spend one day a week doing pretty serious computing, producing wonderful graphics, setting up the mathematical part of it and dumping the whole mess into documents that can be placed on a web page. I am having more fun than I used to, and the students appear to be having a pretty good time while learning a lot. Can all this be bad?"

A very distinguished applied mathematician of my acquaintance is spending part of his time producing cd-roms to publicize his theories and experiences.

The classic modes of elementary and advanced teaching have been amplified and sometimes displaced by computer products. A good computer store has more of these products for sale than there are brands of cheese in the famous Ka-Da-We department store in Berlin. Will flesh and blood teachers become obsolete?

Working habits and the working environment

Is all this perceived as good? Apparently not. Another colleague has written me:

> "On balance, I believe that science will suffer in the multi-media age. My experience is that true thinking now goes against the grain. I feel I have to be rude to arrange for a few peaceful hours a day for real work. Saying no to too many invitations, writing short answers to too many e-mail questions about research. A letter comes to me from a far corner of the world: 'please explain line six of your 1987 paper.' I stay home in the mornings hiding from my office equipment.
>
> Big science projects, interdisciplinary projects, big pushes, aided and abetted by multi-media and easy transportation have diminished my available time for real thought. I am also human and succumb to the glamour of today's technoglitz"

> "
> ***Teachers? Who needs them? The built-in HELP statements, so it is said, can impart plug-and-chug mathematics far more efficiently than any human.***

Is all this really for the better?

Dissemination

Mathematicians used to relegate certain jobs to other disciplines or crafts or professions; we have become our own typists, typesetters, draftsmen, library scientists, book designers, publishers, jobbers, public relations agents, salesmen. For all that the computer is rapid, these activities absorb substantial blocks of time that were formerly devoted to thinking about problems.

In 1597, Tycho Brahe lugged his own heavy printing press from Copenhagen to Prague when he took a job there. We have become matheNETicians.

Teachers? Who needs them? The built-in HELP statements, so it is said, can impart plug-and-chug mathematics far more efficiently than any human.

Journals? Who needs them when you can download papers by the hundreds? The dissemination of mathematics through textbooks and learned journals is threatened in favor of on-line electronic publishing. Every man and woman his own journal. And we may be writing papers that might look like a code or a flowchart, that expose links more easily and prominently.

Whereas people such as Copernicus and Newton waited years before they published, today's scientists, under a variety of pressures, go on line with electronic publishing before their ideas are out of the oven, often unchecked, and they can get equally rapid and equally half-baked feedback.

Self-publication is rife. Refereeing has diminished. The authority that once attached to the printed page has vanished. Are books obsolete?

A collaborator working with me and much in love with mathematical databases, picked up (after clever filtering) more than 100,000 references to a key word that was relevant to our work. This produced an immediate blockage or atrophy of the spirit in him. We wondered whether we could afford the time to assess this raw, unassimilated information overload or simply plow ahead on our own as best we could.

Semioticist and novelist Umberto Eco wrote, in "How to Travel with a Salmon," "...the whole information industry runs the risk of no longer communicating anything because they tell too much."

Nonetheless, my eyebrows were raised recently when I learned that as part of a large grant application to the NSF, the applicants were advised to include a detailed plan for the dissemination of their work. In the multi-media age, mathematics is being transformed into a product to be marketed as other products.

The public understanding of mathematics

Every aspect of our lives is increasingly being mathematized. We are dominated by and are accommodating to mathematical machines and the arrangements they prescribe. Yet, paradoxically, the nature of technology makes it possible, through chipification, for the mathematics itself to disappear into the background and for the public to be totally unaware of it.

It is probably the case that despite the claims of educational administrators, the general population needs now to know less mathematics than at any time in the last several hundred years. What civilization needs is a critical education that brings an awareness and judgement of the mathematics that dominates it, and is able to react with some force to evaluate, accept, reject, slow down redirect, reformulate these abstract symbols that are affecting their concrete lives. Technology is not neutral. It fosters certain kinds of behavior. Mathematics is not neutral.

Individual mathematicians are aware of this. Groups calling themselves "technorealists" have web sites. But with some exceptions, the awareness has not yet penetrated the educational process.

Benefits

To all these developments there will surely be benefits, and the advancing waves of new technologies that are sweeping over us hardly need a public relations agent to trumpet the benefits. The size of Bill Gates' personal fortune derives from the public saying "yes" to all this. "Bring on more." The mental gridlock has only begun to appear. (Emboutaillage mentale.)

In making up the previous list, I have ignored pure fields out of personal incompetence. I simply do not have the knowledge or authority to single out from a hundred expanding subfields the ones with particularly significant potential. We have already heard about many of them in other Urania Talks and in the more than 1400 talks and poster-presentations given at ICM 98.

II

The inner texture (or soul) of mathematics

Let me now go up a metalevel and ask: how will multimedia affect the conceptualization, the imagery, the methodology of future mathematics? The metaphysics or philosophy of the subject? What is mathematics going to be doing other than presenting displays of its own narcissism? This is for me a tremendously interesting but difficult question. This is really what I want to discuss this evening. Here again, all I can do is to describe what I see and to project forward.

I shall make my attempt in terms of what I call

The tensions of texture

The discrete vs. the continuous

This is the point of William Everdell's "The First Moderns." To be up to date, apparently, is to be discrete, discontinuous. Everdell shows how this has operated in literature and art as well as in science and mathematics. Not so long ago, there was a movement afoot in the USA, asserting that continuous mathematics ought to give way in education and in philosophy to discrete mathematics. This movement seems to have quieted down a bit.

The deterministic vs. the probabaiistic

This split applies not only to the modeling of the exterior world, but resides interior to mathematics itself. It relates to such questions as: are the truths of mathematics probabalistic, or to the question of the extent to which we are able to live with such truths. Yet again, one little old lady residing in Rhode Island, winning $17,000,000 in the powerball lottery, does more to question and destroy the relevance of probability theory in the public's mind than all the philosophical skeptics such as myself.

The two dichotomies just mentioned are old, but they persist. The last goes back surely as far as the philosophic discussions of "free will."

And now for some new dichotomies.

Thinking vs. clicking

I have heard over and over again from observers that "thinking increasingly goes against the grain."

Is thinking obsolete or becoming more obsolete? To think is to click. To click is to think. Is this the equation for the future?

Did not mathematician/philosopher Alfred North Whitehead write in one of his books that it was a mistake to believe that one had constantly to think? Do not the rules, the paradigms, the recipes, the algorithms, the theorems and generalizations of mathematics reduce the necessity for thought? Did not Descartes write that his specific goal was to bring about this condition?

And in the historic past was not thinking confined to a special class of people? Have not thoughts over the whole of history been controlled--otherwise one might be declared a dangerous heretic or a traitor? Were not women forbidden to think and to study?

Experimental mathematics, visual theorems, are increasing in frequency. There are now two types of researchers: the first try to think before they compute; others do the reverse. I cannot set relative values on these strategies.

A researcher in AI has written me:

> "Your question 'Is Thinking Obsolete' is very much to the point. This has certainly been the trend in AI over to the past ten years (just now beginning to reverse itself)—trying to accomplish things through huge brute-force searches and statistical analyses rather than through high-level reasoning."

We can also ask in this context, is traditional advanced mathematics obsolete? For example, what portions of a theory of differential equations retain value when numerical solutions are available on demand; when, in many fields, computation is far in advance of success in explaining what is going on by analytic or theorematic mathematics?

Words or mathematical symbols vs. icons

A Semanticist, Mihai Nadin, now teaching in The University of Wupperthal, has written a large book, *The Civilization of Illiteracy*, on the contemporary decline of the printed word; how the word is being displaced by the hieroglyphic or iconic mode of communication. There is now computer induced illiteracy and innumeracy.

There is no doubt in my mind but that this displacement will have a profound affect on the inner texture of mathematics. Such a shift already happened 4000 years ago. Numbers are among the oldest achievements of civilization, predating, perhaps, writing. In his famous book, *Vorgreichischer Mathematik*, Otto Neugebauer "explains ... how hieroglyphs and cuneiform are written, and how this affects the forms of numbers and the operations with numbers."

Another such shift occurred in the late Middle Ages when algebraic symbolisms began to invade older texts.

Mathematics as objective description vs. mathematics by fiat, or the ideal vs. the constructed and the virtual

Applied mathematics deals with descriptions, predictions and prescriptions. We are now in a sellers' market for all three. Prescriptions will boom. There may indeed be limits to what can be achieved by mathematics and science (there are a number of books on this topic), but I see no limits, short of the willingness of humans to endure them, to the number of mathematizations that can be prescribed and to which humans are asked to conform.

"
All products, all human activities are now wide open to prescriptive mathematizations.

In the current advanced state of the mathematization of society and human affairs, we prescribe the systems we want to put in, from the supermarket to the library to the income tax to stocks and bonds to machines in the medical examination rooms. All products, all human activities are now wide open to prescriptive mathematizations. Prof. David Mumford anticipates a great increase in the invention of new mathematical structures. The potentialities and the advantages envisaged and grasped by the corporate world will lead it to pick up some of the developmental tab. And, as it does, the human foot will be asked, as with Cinderella's sisters, to fit the mathematical shoe. If the shoe does not fit: tough for the foot.

What is proved vs. what is observed

This is the philosophical argument between Descartes and Giambattista Vico. I venture that as regards the generality of users of mathematics, its proof aspect will diminish. Remember: mathematics does not and never did belong exclusively to those who happen to call themselves mathematicians and who claim to pursue Mathematics with a capital M. I would hope that the notion of proof will be expanded so as to be acknowledged as simply one part of a larger notion of mathematical evidence.

The whole present corpus of mathematical experience and education has come under attack from at least two different social-political directions:

Euro or Western mathematics vs. other national or ethnic mathematics

We have today's ethno-mathematicians to thank for reminding us that different cultures, primitive and advanced, have had different answers as to what mathematics is and how it should be pursued and valued. (E.g., ancient oriental mathematics was carried on in a proof-free manner. Ancient Indian mathematics expressed itself in verse.) More important than drawing on ancient, "non-Western" material is the possibility that new "ethnic" splits, to be described momentarily, will emerge from within current practices. Will a civilization of computer-induced illiteracy compel major paradigm shifts in mathematics? Extrapolating from Nadin's book, one might conclude that this might arrive sooner than we think and perhaps more rapidly than is good for us.

Male vs. female mathematics

Mathematics has been perceived as an expression of male machismo. Margaret Wertheim is a TV writer as well as a former student of math and physics. Let me quote from her recent book *Pythagoras' Trousers*:

> "One of the reason more women do not go into physics is that they find the present culture of this science and its almost antihuman focus, deeply alienating. ... After six years of studying physics and math at university, I realized that much as I loved the science itself, I could not continue to operate within such an intellectual environment." (p. 15)

The bottom line of this book is that if more women were in mathematics and science (particularly in physics), then they would create

> "an environment in which one could pursue the quest for mathematical relationships in the world around us, but within a more human ethos." ... "The issue is not that physics is done by men, but rather the kind of men who have tended to dominate it." ... "Mathematical Man's problem is neither his math nor his maleness per se, but rather the pseudoreligious ideals and self-image with which he so easily becomes obsessed."

More women are entering mathematics and science, and it will take at least two generations to observe whether or not Wertheim's vision will materialize and what it implies.

The apparent vs. the occult

In a somewhat disturbing direction, we have the concern on the part of some mathematicians and physicists with hermeticisms, apocalypses of various sorts: final theories of everything, secret messages hidden in the Bible, everything under the sun implied by Goedel's Theorem.

I was shocked recently to read that one of the Mathematical Societies in the USA had published some of this kind of material—even though it was in a spirit of "fun."

The old marriage of literacy and rationality, in place since the Western Enlightenment, seems to be ending in divorce. Rationality has shacked up with fanaticisms. Are these part of the breakdown of a literate civilization or merely the age old and temporary anxiety that accompanies the arrival of a new millennium?

Soft mathematics vs. traditional mathematics

I have picked up the term "soft mathematics" from Keith Devlin's popular book *Goodbye, Descartes* which describes the difficulties of the relationship between natural language, logic, and rationality. These difficulties, Devlin asserts, cannot be overcome by traditional mathematics of the Cartesian variety, and he hopes for the development of a "soft mathematics"—not yet in existence—that

> "will involve a mixture of mathematical reasoning, and the less mathematically formal kinds of reasoning used in the social sciences." Devlin adds that, "perhaps most of today's mathematicians find it hard to accept the current work on soft mathematics as 'mathemat-

> ics' at all."

Nonetheless, some see the development as inevitable, and Devlin uses as a credentialing authority the mathematician-philosopher Gian-Carlo Rota. Rota comes to a similar viewpoint through his phenomenological (Husserl, Heidegger) orientation.

After listing seven properties that phenomenologists believe are shared by mathematics (absolute truth; items, not objects; nonexistence; identity; placelessness; novelty; rigor), Rota goes on to say:

> "Is it true that mathematics is at present the only existing discipline that meets these requirements? Is is not conceivable that some day, other new, altogether different theoretical sciences might come into being that will share the same properties while being distinct from mathematics?"

Rota shares Husserl's belief that a new Galilean revolution will come about to create an alternative, soft mathematics, that will establish theoretical laws through idealizations that run counter to common sense.

And what is "common sense?" It may be closer than we think to what George Bernard Shaw wrote in "Androcles and the Lion," "People believe not necessarily because something is true but because in some mysterious way it catches their imagination."

Platonic (Deistic) philosophies of mathematics vs. humanistic philosophies

British analytic philosophy died from "dead-end-itis." Is philosophy, in general, obsolete in today's world? Perhaps, but let us have a bit more of it before we say Nunc Dimittis.

The vaunted and (I think) mythic Unity of Mathematics is further threatened by self-contained, self-publishing chat groups. It was already threatened in Poincaré's day by the sheer size of the material available. The riches of mathematics, without contemplative judgements, would, in the words of Poincaré, "soon become an encumbrance and their increase produce an accumulation as incomprehensible as all the unknown truths are to those who are ignorant."

The classic Euclidean mode of exposition and teaching: "definition, theorem, proof" has come under serious attack as not providing a realistic description of how mathematics is grasped, utilized or created.

Platonism and its various offspring, which have been the generally accepted philosophies of mathematics, have come under serious attack. Here are a few quotes that bear on this.

> "By giving mathematicians access to results they would never have achieved on their own, computers call into question the idea of a transcendental mathematical realm. They make it harder and harder to insist as the Platonists do, that the heavenly content of mathematics is somehow divorced from the earthbound methods by which mathematicians investigate it. I would argue that the earthbound realm of mathematics is the only one there is. And if that is the case, mathematicians will have to change the way they think about what they do. They will have to change the way they justify it, formulate it and do it."
> — Brian Rotman

> "I know that the great Hilbert said 'We will not be driven out of the paradise that Cantor has created for us.' And I reply: 'I see no need for walking in.'"
> — Richard Hamming

> "I think the Platonistic philosophy of mathematics that is currently claimed to justify set theory and mathematics more generally is thoroughly unsatisfactory, and that some other philosophy grounded in inter-subjective human conceptions will have to be sought to explain the apparent objectivity of mathematics."
> — Solomon Feferman

> "In the end it wasn't Goedel, it wasn't Turing and it wasn't my results that are making mathematics go in an experimental direction. The reason that mathematicians are changing their habits is the computer."
> — G. J. Chaitin

III

A Personal Illumination

Here, then, are some of the "tensions of mathematical texture" that I perceive. Today's scientist/mathematician spends his or her days in a way that is vastly different from 50 years ago, even 20 years ago. Thinking now is accomplished differently. Science is now undergoing a fundamental change; it may suffer in some respects, but it will certainly create its own brave new world which will proclaim new idealisms.

I think there will be a widening to what has been traditionally been considered to be valid mathematics. In the wake of this, the field will again be split just as it was in the late 1700's when it split into the pure and the applied. As a consequence, there will be the "true believers" pursuing the subject pretty much in the traditional manner, and the "radical wave" pursuing it in ways that will raise the eyebrows and the hackles of those who will cry "they are traitors to the great traditions."

Elias Canetti, Nobelist in Literature (1981), in his autobiography speaks of an illumination he had as a young man. Walking along the streets of Vienna, he saw in a flash that history could be explained by the tension between the individual and the masses.

Walking the streets of my home town, I got an illumination: the history of future mathematics will be seen as the increased tension and increased interfusion, sometimes productive, sometimes counterproductive, between the real and the virtual. How these elements will play out is now a most excellent subject for writers of mathematical fantasies.

THANKS
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continued on page 37

Mathematicians Can Be Wrong

Tony Dunlop
Ken Kaminsky
Augsburg College

"Here are some quotes from famous mathematicians. While mathematicians are trained to think perfectly logically, we see that they are still fallible."

"Mathematics is the science of what is clear by itself."
--Karl Gustav Jacob Jacobi

Then we can fire all teachers?

"Calculus required continuity, and continuity was supposed to require the infinitely little, but nobody could discover what the infinitely little might be."
--Bertrand Russell

In the 1950's, along came Abraham Robinson's Non-standard Analysis, to tell us once and for all what "infinitely little" means.

"The fact that all Mathematics is Symbolic Logic is one of the greatest discoveries of our age, and when this fact has been established, the remainder of the principles of mathematics consists in the analysis of Symbolic Logic itself."
--Bertrand Russell

Poor Bertrand, missed again! Kurt Goedel's famous Incompleteness Theorems of the 1930's exhibited statements which, while mathematically true, can never be proved logically.

"Mathematics is the cheapest science. Unlike physics or chemistry, it does not require any expensive equipment. All one needs for mathematics is a pencil and paper."
--George Polya

This was actually true until about 10 years ago.

"How dare we speak of the laws of chance? Is not chance the antithesis of all law?"
--Bertrand Russell

The "laws of chance," embodied in the theory of probability, is now so well-established that this statement seems terribly naïve.

"Point set topology is a disease from which the human race will soon recover."
--Jules Henri Poincare

This statement is nearly a century old; topology is now considered an important branch of mathematics, and lies on a foundation of point sets.

Reprinted from "Augarithms," the weekly newsletter of the Department of Mathematics at Augsburg College.

To My Students

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Distributed since 1980

One of the distinguishing characteristics of Harvey Mudd College is the requirement that the study of engineering and science should be accompanied by the study of the humanities and social sciences. This requirement is not only to help HMC graduates to be well rounded people, but also to help you achieve a better understanding of engineering, science and their applications and interactions with the rest of the world. A narrow focus on the technical content will decrease understanding of that technical content. In fact, if the focus is too narrow, it may be impossible to understand anything at all!

The humanities and a humanistic point of view are not extra additions to the technical content, but are part of the technical content. A humanistic point of view permeates the effort to understand all subjects, including mathematics.

The syllabus seems so crowded, that one might assume that there is no room for anything but a narrow focus on technical content, and that a humanistic approach will require that some of the technical content will have to be dropped. I assure you that that assumption is not correct. My experience has shown that, actually, the students learn at least as much, and usually more than other students in more narrowly focused courses. Learning is not effortless or automatic, but the students have found the course to be less stressful and more leisurely.

SCIENCE IS MORE THAN FACTS AND FORMULAS

The facts and formulas are really a small part of a subject, although in high school most class time is spent on them. College is different. Here we are dealing with general principles and how the ideas and concepts fit together. That which is beyond the facts and formulas may be difficult to describe, but it can be discussed in general terms. It seems paradoxical that concentrating on that which is beyond the facts and formulas will give us a greater mastery over those very facts and formulas that have occupied most previous exams.

THE MODERN WORLD

The invention of the Calculus is one of the most important events in the history of civilization. It separates medieval from modern times. Our whole mentality and mode of thinking are decisively influenced by modern science. Newton's ideas set the program for science and social science for 200 years. The success of the quantitative approach suggested to physiologists and psychologists that they look for explanations of their problems in mechanical terms instead of in terms of astrological portents, soul, mind, spirit, humors, and other vague notions. Appreciation of the already amazing power of mathematics and science imbued thinking men with enthusiasm for a sweeping reorganization of all knowledge. They exalted human reason as the most effective instrument for the attainment of truths. Because they regarded mathematical reasoning as the embodiment of the purest, deepest, and most efficacious form of all thought, they urged the use of mathematical methods and mathematics proper for the derivation of knowledge. The chief characteristic of this new approach to knowledge was unbounded confidence in reason and in the validity of the extension of mathematical methods throughout the physical and formal sciences and beyond them to all fields of knowledge (Kline).

MEANS AND ENDS

So learning mathematics brings one right to the heart of things. The subject is intertwined with the translation of perceptions from our senses to science and technology, and with fundamental philosophical and historical questions. We shall study not only the facts, formulas, and techniques, but also the spirit and structure of the subject as one of the humanistic disciplines. The facts, formulas, and techniques will be the means, not the ends; they will be the building blocks for the edifice of science. Our ends or goals will be to comprehend the principles of mathematics and science and to enter into the spirit of creative understanding and discovery.

LEARNING TO QUESTION

The essay "Learning to Question" asserts, "The cutting edge of knowledge is not in the known but in the unknown, not in knowing but in questioning. Facts, concepts, generalizations, and theories are dull instruments unless they are honed to a sharp edge by persistent inquiry about the unknown." The author's advice will be a guide for us. Not only will we ask questions about how to solve a problem or how to prove a theorem, but also such questions as can the solution to the problem be used in any other way; or is this problem special or general; or why is it interesting? Questions about why is a theorem significant and what are the essential ingredients of a proof will concern us. Albert Einstein felt that his imagination and curiosity were more important than his knowledge of certain facts. One of our goals will be the strengthening of our imagination and questioning ability. Another goal, which depends on imagination and questioning, is to learn to approach problems and subjects as experts rather than as novices. We shall become aware of our own growth toward expertness.

PRODUCTIVITY AND UNDERSTANDING

The article from the National Academy of Sciences is about science education. It is also about a real measure of knowledge and understanding. The last page of the article discusses the acquisition of language which distinguishes humans from the other animals. The "...extraordinary output of new sentences by men out of their stock of words and syntax. It is this that makes human language such a remarkably revealing mirror and so potent a tool of the mind and spirit. The same test is most congenial for any real understanding of mathematics and science at every level. What can the student do with what he knows to make a new sentence?" This class will follow the recommendation of the National Academy of Sciences. We shall try to make "new sentences" out of the facts, formulas, and techniques that we learn. Everyone will have the opportunity to be creative by asking questions or finding answers, or by distorting a textbook problem until it is no longer solvable and then studying the difference between solvable and unsolvable problems. There are, of course, many ways to be creative; finding them is an act of creativity.

ROLE OF UNCONSCIOUSNESS

One of the best essays on creativity is *The Psychology of Invention in the Mathematical Field* by Jacques Hadamard. Although the title mentions mathematics, the ideas apply to all fields. There are idea gems throughout the book, and everyone will discover profitable insights by reading it. A central idea is the role of the unconscious in discovery or problem solving. Many people have had the experience of trying to solve a problem, or even to remember something. After much effort the struggle is abandoned; then the solution or answer pops into consciousness at a moment of relaxation.

This surprising situation is common and is described by Hadamard at length. A summary follows:

- First there must be *recognition* of a problem or puzzlement.
- This is followed by *preparation* or study.
- The important third stage is *incubation*. This is the stage when our unconsciousness is working efficiently. The stage of *incubation* is often neglected by students, perhaps because most previous problems were trivial.
- The moment when the solution pops into consciousness is called *illumination* or *revelation*.
- The final stage is *verification*, which is a working out of the details that the sudden insight revealed.

This five stage sequence has been described by many people who have thought about creative discovery or problem solving:

0. Recognition
1. Preparation
2. Incubation
3. Illumination
4. Verification

This scheme has been described by chemists, physicists, novelists, poets, composers, psychologists, and others who create and solve problems. An awareness of the scheme will help in all of your college subjects as well as beyond the classroom.

ROLE OF BEAUTY

G.H. Hardy writes, "The mathematician's patterns, like the painter's or the poet's, must be *beautiful*; the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics."

Hadamard, discussing the general direction of research, asks how the important choice of direction is to be made. He answers emphatically, "The answer Is hardly in doubt: it is the same which Poincaré gave us concerning the means of discovery, the same for the *drive* as for the *mechanism*. The guide...is that sense of scientific beauty, that special esthetic sensibility, the importance of which he has pointed out."

Although the quality of beauty seems to be a personal, subjective concept, general agreement somehow occurs, and many people refer to it with no hesitation. Heisenberg's essay *The Meaning of Beauty in the Exact Sciences* explores the question in a broad sweep from Pythagoras to Einstein. Following the Classical Greek tradition, he defines beauty as the proper conformity of the parts to one another, and to the whole. He considers what motivated Kepler, Newton, Pauli, and Carl Jung. His writing is compelling. He asks, "But are we dealing here with knowledge merely, or also with the beautiful? And if the beautiful is involved, what role did it play in the discovery of these connections?... What is it that shines forth here? How comes it that with this shining forth of the beautiful into exact science the great connection becomes recognizable, even before it is understood in detail and before it can be rationally demonstrated? In what does the power of illumination consist, and what effect does it have on the onward progress of science?"

Freeman Dyson quotes Hermann Weyl, "My work always tried to unite the true with the beautiful; but when I had to choose one or the other, I usually chose the beautiful." [World of Mathematics, p. 1831].

This point of view which is so strongly stated by so many creative people, will be a guide for our own creative and learning efforts.

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"Cantor's Coat"

Lawrence Mark Lesser

In his coat,
Cantor kept a note
his father wrote
Urging him
that faith within
will sustain him.

He picked his path
the freedom of math
but met with wrath
For counting infinities
real as the Trinity
uncountably vast.

Cantor believed this was Divine Plan
But his mentor said, "God made whole numbers,
the rest is by man!"

So his mentor withdrew,
said "Your renegade view
corrupts the youth!"
Kept from his goal,
Cantor searched his soul
to fit part to whole...

"Father, do I hear cries of Galileo?
For this fight, was I right to give up the violin?"

Some say he went mad
from the quest he had
or was he just sad
When his heart went still--
sanitorium swill
against his will.

But in his coat,
they found a note
his father wrote.

A New Family of Irrational Numbers with Curious Properties

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1. INTRODUCTION

The "*metallic means family*" (MMF) includes all the quadratic irrational numbers that are positive solutions of algebraic equations of the type

$$x^2 - nx - 1 = 0$$
$$x^2 - x - n = 0$$

where n is a natural number. The most outstanding member of the MMF is the well-known "golden number." Then we have the silver number, the bronze number, the copper number, the nickel number and many others. The golden number has been widely used by a great number of very old cultures, as a base of proportions to compose music, to create sculptures and paintings or to build temples and palaces (see the first chapter of Reference [1]). With respect to the many relatives of the golden number, a great part of them have been used in different researches that analyze the behavior of non linear dynamical systems when they proceed from a periodic régime to a chaotic one. Notwithstanding, there exist many instances of application of these numbers in quite different knowledge fields, like the one described by the mathematician Jay Kappraff [2] in his study of the old Roman proportion system of construction. This system was based on the silver number, on account of a mathematical property, which is not unique but is common to all members of the MMF, as we shall prove.

Being irrational numbers, all the members of the MMF have to be approximated by ratios of integer numbers in applications to different scientific fields. The analysis of the relation between the members of the MMF and their approximate ratios is one of the goals of this paper.

2. CONTINUED FRACTION EXPANSIONS

The expansion of a real number in continued fractions is one of the most useful tools of arithmetic. Every real number x may be expanded in continued fractions

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

denoted by $x = [\, a_0, a_1, a_2, \ldots]$. The first coefficient can be equal to zero (whenever the real number is between 0 and 1), but the rest of the coefficients are positive integers. The sequence of coefficients is <u>finite</u> if and only if x is a rational number, like

$$\frac{18}{7} = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3}}} = [2,1,1,3].$$

If, instead, x is an irrational number, the expansion is infinite and if we take a finite number of terms like

$$\sigma_k = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ldots + \cfrac{1}{a_k}}}$$

we get a sequence of "*rational approximants*" to the number x that converge to x when $k \longrightarrow \infty$.

Some irrational numbers like π and *e* have approximants that converge very quickly. The number $\pi = [3, 7, 15, 1, 292, ...]$ converges so quickly that the third rational approximant $\sigma_3 = \frac{335}{113} = 3.1415929\ldots$ has six exact decimal figures. Tsu Chung Chi, China, 5th century, already knew this result! Instead $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 2, 2, 8, 1, ...]$ converges more slowly at the beginning, due to the existence of many "ones" in the expansion. The quadratic irrational numbers converge more slowly. "*Periodic*" continued fraction expansions are denoted with a bar over the period, and if the expansion is of the form $x = [\overline{a_0, a_1, \ldots, a_n}]$, we call it a "*purely periodic*" continued fraction. The French mathematician Joseph Louis Lagrange (1736-1813) proved that a number is quadratic irrational if and only if its continued fraction expansion is periodic (not necessarily purely periodic).

Property No. 1 of the MMF

They are all positive quadratic irrational numbers.

In fact, if we solve the quadratic equation

(2.1) $$x^2 - nx - 1 = 0$$

we find that the positive solutions are of the form $x = \frac{n + \sqrt{n^2 + 4}}{2}$. For $n = 1$, we have the well-known golden number $\phi = \frac{1 + \sqrt{5}}{2} = 1.618\ldots$. How do we get its continued fraction expansion? Simply, divide equation (2.1) by x (not zero): $x = n + \frac{1}{x}$ and replace the x of the right member iteratively by $n + 1/x$. In this way, we get after N iterations:

$$x = n + \cfrac{1}{n + \cfrac{1}{n + \cfrac{1}{\ldots + \cfrac{1}{n + \cfrac{1}{x}}}}}$$

If $N \longrightarrow \infty$, then $x = n + \cfrac{1}{n + \cfrac{1}{\ddots}} = [\overline{n}]$, that is, a purely periodic continued fraction. Obviously, the golden number has the most simple continued fraction expansion of all the metallic numbers

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}} = [\overline{1}].$$

Similarly, solving the quadratic equation $x^2 - 2x - 1 = 0$, whose positive root is the silver number, we get its purely periodic continued fraction expansion

$$\sigma_{Ag} = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}} = [\overline{2}].$$

Solving $x^2 - 3x - 1 = 0$, we get the bronze number

$$\sigma_{Ag} = \frac{3 + \sqrt{13}}{2} = [\overline{3}].$$

Summarizing, looking for positive solutions of quadratic equations of the type $x^2 - nx - 1 = 0$ (n natural number), we obtain members of the MMF whose continued fraction expansion is purely periodic $x = [\overline{n}]$. Similarly, the positive solutions of quadratic equations

(2.2) $$x^2 - x - n = 0,$$

(n natural number), are members of the MMF whose continued fraction expansion is periodic $[m, \overline{n_1, n_2, \ldots, n_n}]$. Some of these members are natural numbers of the form $[n, \overline{0}]$. This subset of metallic numbers has curious mathematical properties related to the frequency with which natural numbers appear as well as to the period length or the appearance of "*stable cycles*" (see [3]).

Of all the members of the MMF, the golden number has the most slowly convergent expansion, that is

The golden number ϕ is the most irrational of all irrational numbers.

It is easy to prove that it is sufficient to consider the positive solutions of equation (2.1) and (2.2), since in the remaining cases we have the following results:

a) $x^2 + nx - 1 = 0$. Same solutions as (2.1) but only its decimal part.
b) $x^2 + nx + 1 = 0$. No positive solutions.
c) $x^2 - nx + 1 = 0$. Positive solutions have periodic continued fractions expansions.
d) $x^2 + x - n = 0$. Positive solutions have periodic continued fractions expansions.
e) $x^2 + x + n = 0$. No positive solutions.
f) $x^2 - x + n = 0$. No positive solutions.

3. FIBONACCI SEQUENCES

A Fibonacci sequence is constructed by taking each term equal to the sum of the two precedents. Beginning with F(0) = 1; F(1) = 1, we have

(3.1) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...

where

(3.2) $F(n+1) = F(n) + F(n-1)$.

This sequence may be generalized by taking each term equal to a linear combination of the two precedents a, b, $pb + qa$, $p(pb + qa) + qb$, ... These sequences are called "*generalized Fibonacci sequences,*" (GFS) and they satisfy relations like

(3.3) $G(n+1) = pG(n) + qG(n-1)$

with p and q natural numbers. Dividing both members by G(n) we have

$$\frac{G(n+1)}{G(n)} = p + q\frac{G(n-1)}{G(n)} = p + \frac{q}{\frac{G(n)}{G(n-1)}}.$$

Taking limits and assuming that $\lim_{n\to\infty}\frac{G(n+1)}{G(n)}$ exists and is equal to a real number x (for a proof see Reference [4]), we get $x = p + \frac{q}{x}$ or $x^2 - px - q = 0$, whose positive solution is $x = \frac{p+\sqrt{p^2+4q}}{2}$. This implies that

(3.4) $$\lim_{n\to\infty}\frac{G(n+1)}{G(n)} = \frac{p+\sqrt{p^2+4q}}{2}$$

Let us now verify the following property

Property No. 2 of the MMF

They are all limits of ratios of successive terms of GFS.

Indeed, let us assume $G(0) = G(1) = 1$ and choose $p = q = 1$ in (3.4). The sequence coincides with (3.1), and, as is well known, the ratio of two successive terms of it converges to the golden number

$$x = \frac{1+\sqrt{5}}{2} = \phi = [\bar{1}].$$

If $p = 2$, $q = 1$, the sequence is

(3.5) 1, 1, 3, 7, 17, 41, 99, 140, ...

and similarly we get the silver number,

$$\sigma_{Ag} = \lim_{n\to\infty}\frac{G(n+1)}{G(n)} = [\bar{2}].$$

If $p = 3$, $q = 1$, the sequence

(3.6) 1, 1, 4, 13, 43, 142, 469, ...

gives the bronze number,

$$\sigma_{Br} = \lim_{n\to\infty}\frac{G(n+1)}{G(n)} = \frac{3+\sqrt{13}}{2} = [\bar{3}].$$

If $p = 1$, $q = 2$, the sequence

(3.7) 1, 1, 3, 5, 11, 21, 43, 85, ...

gives the copper number $\sigma_{Cu} = 2 = [2,\bar{0}]$. Finally, if $p = 1$, $q = 3$, the sequence 1, 1, 4, 7, 19, 40, 97, ... gives rise to the nickel number, $\sigma_{Ni} = \frac{1+\sqrt{13}}{2} = [2,\bar{3}]$.

4. ADDITIVE PROPERTIES

If we consider the sequence of ratios of consecutive terms of (3.1)

$$\frac{1}{1},\frac{2}{1},\frac{3}{2},\frac{5}{3},\frac{8}{5},\frac{13}{8},\frac{21}{13},\frac{34}{21},\frac{55}{34},\frac{89}{55},\ldots$$

we know that this sequence converges to the golden number ϕ. Let us now form a geometric progression of ratio ϕ: $\ldots,\frac{1}{\phi^2},\frac{1}{\phi},1,\phi,\phi^2,\phi^3,\ldots$ This geometric progression is also a Fibonacci sequence satisfying condition (3.2), as is easy to prove: $\frac{1}{\phi^2}+\frac{1}{\phi}=\frac{1+\phi}{\phi^2}=1.$

The same happens with the silver number σ_{Ag}, starting with the sequence $\frac{1}{1},\frac{3}{1},\frac{7}{3},\frac{17}{7},\frac{41}{17},\frac{99}{41},\frac{140}{99},\ldots$ that converges to the silver number. Indeed the sequence $\ldots\frac{1}{\sigma_{Ag}{}^2},\frac{1}{\sigma_{Ag}},1,\sigma_{Ag},\sigma_{Ag}{}^2,\sigma_{Ag}{}^3,\ldots$ is a geometric progression of ratio σ_{Ag} that satisfies (3.4) with $p = 2$, $q =$ 1, as is easy to verify by the following equalities:

$$\frac{1}{\sigma_{Ag}{}^2}+2=\sigma_{Ag};1+2\sigma_{Ag}=\sigma_{Ag}{}^2;+2=\sigma_{Ag}{}^3;\ldots$$

The same procedure can be applied to every member of the MMF and we may assert

Property No. 3 of the MMF

They are the only positive quadratic irrational numbers that originate geometric progressions that simultaneously satisfy additive properties.

This curious property endows all the members of the MMF with interesting characteristics that become a base of different proportion systems in design.

5. QUASICRYSTALS: FORBIDDEN SYMMETRIES

Among the many physical, chemical, biological and ecological problems in which the members of the MMF appear, one of the most striking is the study of a quasicrystal structure. "*Crystals*" are the most regular, periodic and symmetric of all the real entities. On the opposite edge, there exist the most disordered or amorph configurations: the "*glasses*." How do we distinguish between a crystal and a glass? The answer is very simple: you may model a real crystal by putting an atom or a molecule at every vertex of a regular triangular, square or hexagonal lattice that enjoys symmetry of order 3, 4 and 6, respectively. In such a way, the problem of the structure of matter becomes one of pure geometry. This was the situation until 1984, when Shechtman et al. [5], [6], registering electron diffraction patterns in a rapidly cooled metal alloy, found a fivefold rotational symmetry when making projections with an angle equal to the golden number (see [4]). This new solid state of matter was called a "*quasicrystal*." As is well known, it is impossible to tile a plane using only shapes that have fivefold symmetry. But the two-dimensional analogue of quasicrystals – Penrose tilings – has this symmetry. It is interesting to note that the golden number arises in this tiling. The tiling is made of two types of parallelograms, and in an infinite Penrose tiling, the ratio between the numbers of units in these two types is the golden number.

Starting with this experimental discovery, there appeared experimentally new quasicrystals with other forbidden symmetries. For example, the silver number $\sigma_{Ag} = 1+\sqrt{2} = [\bar{2}]$, generates a quasicrystal enjoying a forbidden symmetry of order 8 (see [7], [8]), while the number $[\bar{2}] = \phi^3$ appears in another forbidden symmetry of order 12 (see [9]).

In particular, Gumbs, Ali et al., in different papers ([10],[11]), have analyzed electronic, optical, acoustic and superconducting properties of quasi-periodic systems. The research consisted of the construction of one-dimensional models of new types of quasicrystals, designed taking as a module the different members of the MMF. They were very interested in these quasicrystals, due to many important physical applications, like light transmission through a multi-layered medium. Among their most prominent experimental results, they found big differences in the behavior of metallic numbers whose continued fraction expansion was purely periodic (the golden number, the silver number and the bronze number) and those metallic numbers whose continued fraction expansion was only periodic (the copper number and the nickel number).

6. CONCLUSIONS

In analyzing, from a mathematical point of view, the similarities as well as the differences among the members of the MMF, it is obvious that these characteristics are strongly linked with the transition from periodic to quasi-periodic dynamics. But simultaneously, from the beginning of humanity, there have been philosophical, natural and aesthetic considerations that have given them primacy in the establishment of geometrical proportions based on some members of this family. Such a broad range of applications opens the road to new multi-disciplinary investigations that undoubtedly will contribute to clarifying the relations between art and technology, building a bridge that should join rational scientific thinking with aesthetical emotion. Hopefully, this new perspective could help us to confer on technology, from which we depend every day more and more for our survival, a more human character.

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The Prospects for Mathematics in a Multi-Media Civilization

continued from page 28

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X

The Möbius Metaphor

Julian Fleron
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When the topic of mathematics is brought up, people associate it with balancing their checkbook, the pressure of memorizing times tables in elementary school, or the misery of high school geometry proofs.

I contend this is not what real mathematics is about.

Instead, I believe mathematics embodies human creativity and intellectual struggle, provides a language to describe and explain the physical world, epitomizes aesthetic beauty as much as art or music, is critically useful, and permeates all aspects of our existence.

To illustrate these last three points, consider the following small example from my mathematics courses for elementary school teachers and liberal arts students:

- Tear off the left edge of this paper as indicated.

- Bring the ends together to make a loop-like those children use to construct Christmas chains.

- Instead of joining the ends as usual, flip one over so the labeled ends are glued together. You have created a Möbius band, named after the German mathematician August Ferdinand Möbius.

So what? Well, this seemingly simple object is in fact quite exotic. For it is one-sided! It is a "non-orientable surface" in mathematical jargon. Check for yourself. Try to color just "one side" blue. Can't be done. In the same way, the Möbius band has only one edge. Moreover, if you try to cut the Möbius band down the middle, or into thirds, well... Look out! All kinds of surprising results await you.

This kind of beauty is the norm in mathematics, not the exception as most believe. It's deep, but not complicated. I share Möbius bands with my future elementary teachers in the hope that their students will construct all their Christmas chains from Möbius bands. For there's no reason an art project, Christmas chains for example, can't be mathematical. As the nineteenth century mathematician Gosta Mittag-Leffler said, "the mathematicians best work is art, a high perfect art."

Nice, but does the Möbius band really matter? You'd be surprised. Because Möbius bands have only one side they make perfect conveyor belts--they wear twice as well! B.F. Goodrich Company patented conveyor belts made in the shape of Möbius bands long ago, and such belts are commonplace in every large industry and machine shop--even today. Discovered in 1983, the chemical compound tetrahydroxymethylethylene, takes the natural shape of a Möbius band. Möbius bands have also provided a model for high performance, nonreactive electronic resistors.

Applications such as this are the norm rather than the exception.

Whether considered on aesthetic or utilitarian grounds, mathematics is ubiquitous. We're not just talking of the arithmetic many confuse with mathematics, but also geometry, pattern, structure and form.

Not convinced? Let's look to the Möbius band again. The Mobius band is the symbol for recycling 1, and recycling's mascot is a Möbius-faced creature appropriately named Möbius. Recycling's Möbius symbols are found everywhere. Suppose each time you see one of these symbols, you look around and see where you can find mathematics. You'll be surprised how much you see. As you look, day after day, you should begin to appreciate the words of the physicist James Hopwood Jeans who remarked, "the Great Architect of the Universe now begins to appear as a pure mathematician."

If all this fails, come join me and my students on our voyage of mathematical discovery. We'll help you find the way.

X

Mathematical Fiction

Pat Mower
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The course History of Mathematics provides an excellent forum for students and educators to explore different approaches to understanding mathematics and the development of mathematics. During the fall of 1996, students enrolled in this course were asked to consider the aesthetic and literary qualities of mathematics and mathematics history through their creations of fiction. The assignment and three examples of their efforts follow.

ASSIGNMENT:
Write a piece of mathematical fiction (poem, short story, play, letter to a mathematician, etc.) that uses some of the themes or a theme or mathematician explored or that will be explored later in this class.

GRADING:
Content/accuracy/relevancy (40 points)
Creativity/effort (30 points)
Mechanics/clarity (30 points)

A Day in the Life of Diophantus (c. 251 A. D.)

Rebecca Pekrul

"This is Good Morning Empire, coming to you from our studio in downtown Rome on this very sad day in A. D. 251. Earlier this week we were all shocked and saddened to learn of the death of Emperor Decius during a battle near the city of Dobruja. We will hear more on this sad story later in our broadcast.

Also, coming up we will have a report from the Weather Desk where Seeum Clearly is watching the sky for us, and later we will go to our Man on the Street reporter, Gofar A'Field, for the latest of his reports on "Future Historical Figures," but first to our news desk where Telly True has the year's top stories."'

"Thank you, Talkum Muchly. Our top story this year is the shocking news of Emperor Decius' violent death near Dobruja, where his troops have been valiantly repelling yet another invasion by those vicious Goths. These barbarians have been sacking and pillaging in the region again despite numerous pay-offs and a negotiated cease fire accord.

"Decius was in the area to encourage regional officers to take a stronger military stand against this type of incursion on the Empire. There are rumors of disloyalty and intrigue flying in all levels of the military this morning.

"Senate investigators are on the scene but are not speaking to reporters because some of the rumors and allegations involve high ranking officers. The person whose name is most frequently mentioned by anonymous sources close to the investigation is, in fact, Gaius Vibius Trebonianus Gallus, who is the highest ranking military officer on the scene. In fact, he is one of the principal candidates for the next Emperor.

"Members of Gaius' personal staff have had no comment, except to confirm that they believe that Gaius will announce his decision on whether to accept the office of Emperor at an appropriate time in the not too distant future. Private sources close to the Citizen believe he will assume the office under the official name E. Valerius.

"Reaction from the Senate is as divided as that august body was during Decius' confirmation hearings just two years ago. Those who were supporters of E. Phillip I are as gleeful as they can be on such a disturbing occasion as this, feeling that this is his just desserts for having been proclaimed Emperor by the military while Phillip was still living. Certain Senators have been heard to mumble “He who lives by the sword, dies by the sword.”

“Those Senators who were less influenced by Phillip have expressed feelings from weary disbelief to total outrage at this affront to the power of Rome and the nature of this august body.

“In the larger community there are many who have expressed relief at the Emperor’s death. A group of persons previously identified as Christians reminded our reporter of the martyrdom of Pope Fabiun last year, the exile within the past year of such persons as Cyprian, Bishop of Carthage, and the imprisonment and torture of Origen, a leading churchman. Members of the formerly Christian group, sometimes known as lapsis, or lapsed Christians, have hopes that they will be able to resume their previous religion without fear of the next Emperor. Those who accepted exile rather than bow their knee to Rome have harsh words for those who publicly repudiated their beliefs under the threat of Roman punishment. Many knowledgeable sources believe there will be a great deal of controversy within the so-called Christian community regardless of the beliefs and actions of the next emperor.

“In other areas of the empire, the Parthians are taking advantage of the continuing chain of frequent deaths and apparent weaknesses in Roman leadership to make threatening moves toward our borders. At present we have not seen any major cities fall, but some feel that can change rapidly in the near future if our military and civil governments don’t develop a more unified front. The military is asking for additional revenues to pay for border defenses in all areas, while the Senate is complaining of falling tributes and a lack of new slaves in recent years.

“In other news the Agricultural Minister of the Nubian Empire has announced that this year’s crop outlook is very poor. “The soil of this region is not able to maintain the current level of agricultural demand. Regional famine is a very real danger in the near future if the Gods do not hear our prayers and intervene soon.” There have also been some reports of attacks on the trade routes through Nubia. Rival traders may be responsible for the attacks, but so far only a few nomads have been identified as the culprits.

“From the port of Adulis in Ethiopia there are good market reports today. Several caravans have arrived with iron. gold, rhino horn, assorted animals and slaves. Also in port are ships from several trading partner countries with cargoes of weapons, wine and cloth. The Harbor Master at Adulis says trade is brisk, and that more docks and piers are available at very attractive rates. Call for details.

“We now take you live to Alexandria, Egypt, where our Man on the Street Reporter, Gofar A’Field, is waiting to show you that wondrous city and one of its leading citizens. Gofar, can you hear us?”

“Yes, thank you, Telly. As you can see, I am standing on the edge of the Red Sea in the Port of Alexandria. It is a very well planned seaport, as one would expect of a city built by the order of Alexander the Great. The streets are straight, and laid out in legion like grids with elegant colonnades on the principal avenues.

“On the east end of the harbor is the upper class section of the city, known as the Brucheim. This is where the aristocracy built their magnificent homes during the time of Alexander and the Ptolemies. Today some of the most elegant homes in the area are occupied by Roman administrators and military leaders.

“As we continue along the harbor you can see the Serapeion and then the Soma. which is the mausoleum of Alexander the Great and the Ptolemies. Next is the Temple of Poseidon and the Museum/Library complex. The final buildings of note are the Theater and the Emporium Marketplace, where most of the social and commercial life of the city is conducted.

“Our destination today was a small house near the Library where we dropped by the home of the noted mathematician Diophantus. There we were privileged to observe a normal day in the life of this “Future Historical Figure.” We will now roll the tape which was shot earlier in the home of our subject and his lovely wife.”

(Roll Tape)

Wife: “Diophantus, I’m headed to the market. Is there anything that you need?"

Diophantus: "Yes, I need more paper. Would you pick me up another papyrus or two?"

W: "What do you mean, another papyrus? You just got a whole new papyrus a week ago. What did you do with all of that?"

D: "Well, don't you remember that Anatolius was here Monday, Wednesday and Friday?"

W: "So?"

D: "So, we were working on Book I of *Arithmetica*."

W: "Oh, that again!" When are you going to quit adding to that stu, ah, book of yours? Aren't you ever going to quit wasting paper on that?"

D: "What do you mean, 'wasting paper,' woman? Don't you know that the whole world is waiting impatiently to see my latest mathematical solutions and discoveries in number theory?"

W: "The whole world, huh? Just how soon are they all going to beat down our door to pay for a copy of this magnificent scroll? I still haven't seen anyone except the librarian at the Serapeum ask for a copy, and that's just because he's supposed to have a copy of everything written!"

D: "That's not true, Bishop Dionysius wants a copy too."

W: "Sure, just because you dedicated it to him! Which, by the way, was a very politically incorrect move if you ever wanted to go on a Roman Holiday, since Emperor Decius has been feeding Christians to the lions!"

D: "Well, even so, wife, what I'm doing is important even if there aren't many people who appreciate it. How many copies of Euclid's *Geometry* do you suppose were made the first year or so after it was written?"

W: "So you're saying that this *Arithmetic* of yours is as important as Euclid's *Geometry*?"

D: "Yes, I am! Nobody has ever done much in the way of solving two quadratic equations at once. The Egyptians and Greeks up to now have just been messing around with a combination of linear and quadratic equations, but I'm the first to work with two quadratics simultaneously!"

W: "How many of these wonderful problems are you planning to record for posterity?"

D: "Well, I haven't entirely decided yet, but 130 seems like a good number."

W: "How many papyri is that going to take?"

D: "Oh, somewhere around 13."

W: "Thirteen! That's preposterous! Who's ever heard of one man writing 13 papyri full of arithmetical nonsense?"

D: "What's wrong with thirteen? I'm sure Euclid wrote at least that many *Geometry* books."

W: "So? You think you have to use enough papyrus to wallpaper the whole house (as if this place were big enough to compare with a mouse hole) just to write as many books as some Greek geometer who's been dead for 500 years!"

D: "Well it takes a lot of paper to write out each problem in its entirety and then solve the problem in precise terms. I have to make it clear what I'm about. It wouldn't be a very good text if the reader couldn't easily understand what was written."

W: "What do you mean write out the problem? Show me an example."

D: "We wish to find two square numbers such that the excess of the larger over the smaller. when subtracted from the larger, leaves a square number, and also, when subtracted from the smaller, leaves a square number."

W: "Well, isn't there a shorter way to write it?"

D: "NO!"

W: "Oh. there has to be. See, here's my shopping list."

<She writes on board.> 1 pp/M, 15 poms, 3 pt o.oil, 5# om, 5# che/EmC; 5 yd wht lin, 8 yd gy cot /WRM; see AG/bbsit Wed., ret/Ms rpin.

D: “So, what’s that supposed to mean?”

W: “It says: one papyrus from Millitonius’ Officius Maximumis. Fifteen pomegranates, three pints of olive oil, five pounds of oatmeal, and five pounds of cheese from Emporium Centrius. Five yards of white linen cloth and eight yards of gray cotton cloth from West Ridge Mallius. See if Annalisa Galivantius will baby sit while I go to Alter Guild meeting on Wednesday. Return Matilda’s rolling pin. Now, how would you feel if I wrote that all out in complete and logical sentences? I’d use paper as fast as you do, instead of just a scrap now and then!”

D: “I see what you mean. It is a bit shorter your way. Let me work on this a bit. I guess instead of writing ‘the unknown quantity’ I could write ‘x’ and instead of ‘the unknown quantity squared’ I could write ‘x^2.’”

W: “Well that’s a start, but I hope you can improve on it some more.”
(END TAPE)

“This has been a ‘Future Historical Figure’ report, live from Alexandria. Egypt. I am Gofar A’Field saying good bye for all of the Good Morning Empire staff and reporters. Stay tuned for your local news.”

Limericks on the “Century of Genius”

Thomas J. Lipp

In the century of genius they reigned,
When great strides in Math were gained.
From 16- to 1720
Arose brilliant minds o’plenty,
‘Though things ‘tween church and reason were strained.

Galileo (1564-1642)
Galileo found great hope
In improving the telescope.
With Rome he disputed,
When Ptolemy he refuted,
But, later, acquiesced to the Pope.

Kepler (1571-1630)
Kepler was in a nice groove,
When he saw the planets move.
He caused great commotion
With his laws of their motion.
For Ptolemy they did disprove.

Descartes (1596-1650)
A dreamer and egotist was Descartes,
Who gave Cartesianism its start.
Expanding all knowledge from
His “Cogito ergo sum,”
He took all opposition to heart.

Fermat (1601-1665)
Probability’s founder was Fermat.
Far from calculus he was not.
But he was socially unsmart
And ticked off Descartes,
‘Though it was his advice he sought.

Pascal (1623-1662)
Pascal. Fermat’s false friend.
Much time with vacuums he would spend.
His math machine we adore,
The triangle, press, and more.
He was lost to religion in the end.

Newton (1642-1727)
Newton was a prideful man,
Thinking calculus alone he began.
‘Though his theories on light
Were not perfectly right,
Discovering gravitation was grand.

Leibnitz (1646-1716)
Towards Leibnitz was Newton quite spiteful.
Who was calculus’ founder rightful?
Pick whomever you want.
Leibnitz influenced Kant,
And his theories of monads were insightful.

Bernoulli (1654-1705)
Notable for calculus were truly
James and brother John Bernoulli.
Also Daniel, John's son,
Was a capable one.
The conservation of energy he proved coolly.

Those mathematicians reigned supreme,
Who had the courage to dream.
Had I my druthers,
They'd all be my brothers,
And I'd be on a winning team.

News Extra: Ancient Document Discovered

"Could be the greatest find of all time if validated" says professor

by Shawn Dolezilek
The Associated Press

Members of the Archeology Department at Kansas State University held a news conference yesterday, describing the first translations of an ancient document discovered six months ago at a dig site just outside of Alexandria, Egypt.

"We knew we had something special as soon as we dug it up, but we wanted to be absolutely positive of the translation before we went public," said Edward Wycliffe, head archeologist of the dig which found the document.

The document itself has been carbon dated to 250 A.D., and the author claims to be the ancient mathematician Diophantus. In the document, he describes a "dream" he had about a particular mathematical problem. The extraordinary part of the document, and the controversial part, is the particulars of the dream. These first translators find evidence of specific sequences where he appears to be describing several modern concepts and appliances.

"If this holds up, these descriptions make Nostradamus look like a side-show fortune teller," says prophecy expert Dan Drieffen.

The panel at the news conference noted that many copies had been sent to translation experts and the original copies to the Smithsonian for further scrutiny.

"We have no doubts our results will be validated," said Wycliffe. "And, when they are, we will be forced to take a serious look at ideas about the nature of time that we have previously found laughable."

THE DOCUMENT

Even now, these many years after I was smitten by my dream, it comes back to me as if I were still there. The otherworldly sights and putrid smells of my dream would have been enough to scare lesser men to death, but what I learned from my dream scares me even more. I have forced myself to call what happened a dream, for if I do not, the logical implications would have driven me to madness long ago. Or, at the very least, if I had tried to describe this scene to my contemporaries, I surely would have been laughed out of the Museum. Lying here, knowing now that I have perhaps days, or maybe mere hours left to live does not lessen the sense of fright and wonderment I felt, oh, those many years ago.

I write this text more as an open letter to myself, in an effort to exorcize the memories of that day before I go forth on the Great Journey. After I have finished this and have died, my servants are to bury this a goodly distance from my wonderful city of Alexandria. I hope that time will claim this letter before it is ever found, but if any sons of the Roman Empire, many generations hence, find this work and wonder from whence it came, I am so obliged to tell who I am, and why my story is credible.

I am Diophantus of Alexandria, respected scholar and mathematician of the Museum at Alexandria, the same Museum the likes of the great Euclid and Eratosthenes taught at during the days of Greek primacy. Surely, my contributions to mathematics have been important enough that, provided this letter is ever found, my name will make my tale credible.

Even now, some fifty years after the fact, I still try to make sense of all that I saw, but the fact remains: even my powers of perception and intellect are insufficient to make sense of it all. Therefore, I will attempt to tell my story exactly in the way it happened to me.

It was a typical midafternoon, not unlike any other in my home city. I had decided to go for a walk after a lecture at the museum. I was thinking about a problem of dividing a square number into two other square numbers. I was almost there, and through considerable trial and error, I was ready to make the problem work.

Walking across the Heptastadion, to the island of Pharus, I planned to travel out to the Pharus Turris, where the great lighthouse stands. I had made many trips to that tip of the island. Standing on a landing in the lighthouse, high above the ground, was an awe inspiring sight. Gazing out into the azure sea and watching the ships come into port, I had many insights into the great Mystery that is number. Even now, that landing surely still bears the chalk marks of my fervent computation.

I loved to run up the stairs to that top landing at full speed. I had done this so many times that I was sure I knew every worn stair by heart, or so I thought. This day, as I ran up the stairs, nearly to the landing, apparently I caught a toe on one of the steps. As my body was flung headlong through the air, I felt as a bird must feel. But about the time I decided my lack of connection to the ground was a problem, I saw stars, then blackness surrounded me ...

I thought I had regained consciousness, but I was lying face down in an expanse of wet grass. This contradiction went straight to my brain; I was afraid that I had died and gone to Elysian Fields. But in short order, I had taken inventory of my body and the large lump I felt on the back of my head convinced me that I was indeed still alive.

Sitting up, my eyes scanned my new environment, trying to assess exactly where I was and what had happened to me. What I saw before me was indeed shocking. I was sitting in the middle of a large, well-manicured lawn surrounded on three sides by large buildings, which were not of Greek or Roman design and were in fact quite peculiar. They were all large buildings, each more than three stories tall, and all shone in the sun with a brilliance that made my eyes hurt.

The last side of the field was bordered by a road. But on this road were things I have yet to comprehend. These things were traveling up and down the road at great speeds, greater than even the swiftest horse. I know they were inanimate because I could see people sitting in them, controlling the machine's movements. They were bright and shiny, like the buildings, but the wind blew a putrid and frightening smell from that direction, so I decided not to investigate further.

I decided to make my way to one of the buildings. I picked the largest, assuming I would be able to find some dignitary or scholar that would be able to explain to me where I was and perhaps what had happened to me.

As I approached the buildings, I began to encounter the people of this strange place. Their clothing was like nothing I had ever seen, even in the Nubian lands to the south of Egypt. Their clothing was very bright and colorful, and nearly all of them were carrying some type of packs on their backs. I also noticed the women of this land wore extremely revealing clothing.

Coming closer, I realized their speech was utterly foreign to me. When the first group I approached recognized my presence and turned to look at me, I became scared. What would these strange people do to me? Would they kill me on the spot, or take me into slavery? Mustering all my strength, I put on a smile and strode past them. Luckily, they did nothing. They watched me pass, and then went back to conversing in their strange language.

I could not explain how, but as I made my way through the crowds of people, their language became more and more understandable. I decided this had to be a dream, so I spent no time wondering how I could suddenly understand their speech.

Passing through the doors and into the building was quite a shock to me. It was quite warm outside, but when I entered the building, I was hit with a blast of cold air. It was amazing, hot outside and cold inside. However, it was no more amazing than anything else that had already happened.

Wandering down the hallway, I entered a room that I instantly recognized. A large room, going down into the ground as one approached the front. A large table at the head of the room, and many rows of chairs to sit in, all facing the front. This had to be a lecture hall! All of the people outside all appeared to be young adults, they had to be students! Perhaps my dream would finally start to make some sense.

People began to file into the room, taking their seats and conversing about parties and classwork, topics not unfamiliar in my own lecture hall. I took a seat in the back of the room, deciding to sit and listen to the lecturer. I was interested in seeing what this dream lecturer was going to present.

An older man entered the room, carrying a large stack of papers with him.

"Good afternoon," he said. "Today, I am going to present Diophantine equations to you, paying particular attention to one example we will examine with our graphers."

I was shocked and amazed to hear my own name...what in the name of Alexander himself was going on?

"Diophantus spent most of his time exploring what we would call systems of equations," the lecturer said. "Using our modern methods, we can set up the problems he looked at with two related equations, and, with substitution and algebra, solve for the variables, or find a general case solution for any possible set of variables."

Even though it has been nearly 50 years since my experience, I still remember these words and the rest of the lecture as if it were yesterday, and I am still trying to understand most of it.

"To get a better idea of what I am talking about, we will look at one particular problem he dealt with, namely 'to divide a square number into two other square numbers.' Does that sound like an equation to anyone?" asked the lecturer.

One of the students raised their hand, and said, "Well, how about $x^2+y^2=b^2$?"

"Exactly," said the lecturer. "So you pick any square number you want for b^2, then find a way to solve for x and y, right? Well, the trick is figuring out how to solve for x and y.

"What Diophantus found was that another equation was necessary to solve this problem, and he found that equation to be $y=ax\pm b$, where the b is related to the b^2 in the first equation, and a can be any integer. Of course Diophantus himself didn't know about negative numbers, and any equation that would end up with a solution that would have been negative, he called impossible."

A student raised his hand and asked, "So, how did he come up with that second equation?"

"Trial and error. Lots of it. We have no evidence to prove that this came from any other source than lots of trial and error. However, when we look at this system on our graphers, I think the reason for the choice will become obvious.

"I think, given the two equations, you should be able to solve this problem. So, for next time, pick a square number for b^2 and work out the solutions. But, for right now, let's go ahead and pull out our graphers, and we'll see what this system looks like. I think you'll see how and why it all falls into place."

With this, all the students reached into their packs, pulled out small boxes and began manipulating them in some fashion. The lecturer was doing the same, but he was also doing something with another piece of equipment as well.

Suddenly, the lights went out and the machine the lecturer was working on spit out a huge beam of light against the far wall of the room. On the wall, I could see shadows on the wall that made out words and pictures.

"Okay," said the lecturer. "What do we want for our square number? How about we use one? Okay, now what does the first equation look like?"

"It's a circle."

"That's correct. What about the second equation?"

"A line."

"Right again, but what are we going to pick for the a in the equation? Let's go ahead and pick one as well. What's the b going to be in the second equation?"

"One."

"Right, oh, also let's use the minus to begin with. So, adjusting for how we have to enter the equations into our graphers, we have these two equations:

$$y=\sqrt{\left(1-x^2\right)}$$

y=x-1

"So, what does this look like graphically?"

The lecturer began manipulating the machine again, and the following picture was drawn on the wall. (Figure 1)

"Now, what does this tell us? Well, just looking at the first equation, I would say intuitively that we can see that x=0 and y=1 or -1 or vice versa. Well, that gives us four fixed points, where our circle intersects the x and y axes at four places. Now, for our second equation we picked a=1, so we should see a line with a slope of one, and since we picked minus b, the line should pass through the point (0,-1), which it does. But a line passes *through* a circle at two points, so what's the other intercept? In this case it's (1,0), which we also know is a solution. What if we pick a different number for a? Will all this still hold true? Let's say a=4, make the change in your graphers, but leave the graph of the other line there as well."

Again manipulating the machine in front, the picture changed to look like this: (Figure 2)

"Now," he said, "go ahead and trace the intercept point for this second equation and give me the coordinates. (0.48085, 0.87680) Ok, so now take these numbers, square them, and add them together. What do you get? 0.9999949625. Well, that looks pretty close to one to me. I think that should let you better see the relation between these systems of equations and their graphs. Also, all of our work here was done in the first quadrant, just as all of Diophantus' work was. But, even restricting ourselves to the first quadrant, we still have an infinite number of solutions, so you should be able to see why these are called 'indeterminate equations.'

"Well, that's about all the time we have for today...I'll see you next time."

With that, the lights came on. Apparently, the lecturer saw me for the first time. Pointing at me he said, "You, you aren't enrolled in this class, are you?"

Everyone in the class suddenly turned and looked at me. Someone asked me where the toga party was. Afraid I had broken some taboo against attending lectures in inappropriate dress or some other infraction, I turned and ran out of the lecture room, down the hall, and out of the building. Unfortunately, as I was running down the steps, I tripped on my toga and fell headlong to the ground. Again, the blackness surrounded me...

"Master Diophantus...Master Diophantus," was the next thing I heard. When I opened my eyes, I was again in the lighthouse, lying on the landing. A young servant was standing over me. He had been sent out to search for me, since I had been expected at dinner several hours earlier.

I pushed all my thoughts to the back of my mind and accompanied him back to the Museum. I spoke of this tale to no one, and soon discounted the whole thing as a dream or hallucination. But, I never was able to explain why I had bumps on the back and front of my head, or how I suddenly solved the division of a square into two squares problem.

--Diophantus of Alexandria

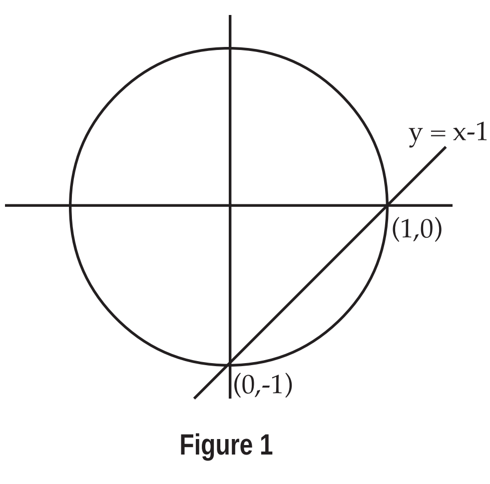

Figure 1

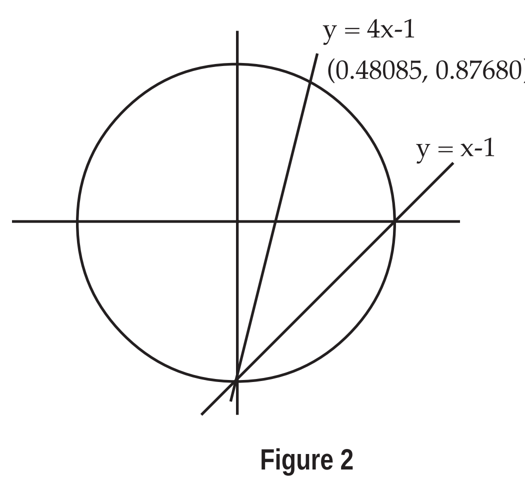

Figure 2

Leibniz: His Philosophy and His Calculi

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This paper is about the last person to be known as a great Rationalist before Kant's Transcendental Philosophy forever blurred the distinction between that tradition and that of the Empiricists. Gottfried Wilhelm von Leibniz is well known both for the Law which bears his name and states that "if two things are exactly the same, they are not two things, but one" and for his co-invention of the Differential Calculus. It is commonly taught that Leibniz and Isaac Newton each independently discovered means to find:

1) the tangent to a curve at a point.
2) the length of a curve, the area of a region, and the volume of a solid.
3) the maximum or minimum value of a quantity.
4) the relation between the velocity and acceleration of a body at an instant and the total distance traveled by that body in a given period of time.[1]

These new mathematical techniques supplanted those of the ancients and provided modern scientists with tools which enabled their science to leapfrog classical science. Since these discoveries were so important to the natural sciences and the ensuing technological development, great prestige came to be attached to them. Competition for this prestige resulted in a bitter dispute in which Leibniz was charged with plagiarism. To chronicle the charges and counter-charges would be an interesting task--but one better left to historians of mathematics.

The 17th century was buzzing with discovery and humming with intellectual activity. Someone would have discovered the calculus before 1700 if neither Leibniz nor Newton had been born.[2] While it is generally agreed that their discoveries were independent, it is known that Newton's discovery preceded that of Leibniz. Though Newton was first, that the centuries have decided in favor of Leibniz's notation because of the relative facility of its use, is testimony to the central position of notation in Leibniz's thinking. For him, thinking is the manipulation of the symbols of notation-facility of manipulation is facility of thought. Anyone who has tried to calculate simple interest using Roman Numerals knows well the importance of an elegant notation.

In the preface to his translations of *The Early Mathematical Manuscripts of Leibniz*, J.M. Child maintains that "the main ideas of [Leibniz's] philosophy are to be attributed to his mathematical work, and not *vice versa*."[3] The esteem in which Leibniz held an elegant notation (the most elegant being the simplest possible way of handling all the possibilities) is all that is offered in support of this. If, by 'main ideas' Child means the form of Leibniz's analysis—that is, that part which has its source in the method which he employs—I would not disagree. But we must ask more than how it is that he performs his analysis, we must also ask what it is that he chooses to analyze. Neither Leibniz's interests nor his optimism knew any bounds. We must remember that in addition to being a great mathematician and logician, he was also the Dr. Pangloss ridiculed by Voltaire in Candide. If recursion of notation is Leibniz's method, *philanthropia* is his motivation. The two are tied together by his views of God and the *philothea* appropriate to the wise man.

The function of God in Leibnizian metaphysics is to be the creator of the universe (the first cause) as well as the source of perfection and order within it (the final cause). Hence, God is, by definition, the perfect creator of the universe. Likewise, by definition, composite substances (bodies) are composed of simple substances[4] and simple substances are unities.[5] Leibniz calls these unified simple substances 'monads' after the Greek *monas*. Since monads are simple, they have no parts. Change occurs when parts are combined together or cleaved apart. "Now where there are no parts, neither extension, nor figure, nor divisibility is possible."[6] Thus monads are changeless, eternal, and by all outer appearances, identical. But the law that bears Leibniz's name tells us that they cannot be identical—for if they were they would be one and the same thing. The dilemma is solved by letting them differ internally. These internal differences lie in the different perceptions and appetitions of the different monads.[7] It is assumed that every monad is subject to change and that change is continuous.[8] But these internal changes "*are inexplicable by mechanical causes*."[9] Mechanical causes apply only to bodies (composite substances) "[T]he perceptions in the monad spring one from the other, by the laws of desires [*appetits*] or of the *final causes of good and evil*, which consist in observable, regulated or unregulated, perceptions; just as the changes of bodies and external phenomena spring one from another, by the laws of *efficient causes*, that is, of motions." He claims that this entails that "there is a perfect harmony between the perceptions of the monad and the motions of the bodies, preestablished at the beginning between the system of efficient causes and that of final causes."[10]

Surely a perfect creator would create the best possible product if not a perfect product. The best possible universe is that in which "there is the greatest variety together with the greatest order;...the most results...; the most of power, knowledge, happiness and goodness in the creatures that the universe could permit."[11] This universe is a plenum—that is "*nature never makes leaps*"[12] —and because of this "everything is connected and each body acts upon every other body..."[13] Bodies are connected together by efficient causes. God, as the creator of the universe, is the only being with a complete knowledge of it. That God does have perfect knowledge of the universe is required by the principle of sufficient reason: "nothing happens without its being possible for him who should understand things, to give a reason sufficient to determine why it so and not otherwise."[14] Other monads perceive the whole universe with some degree of confusion. Some of the least confused are aware that they are representing the perceived objects of the universe to themselves. Leibniz anticipates Kant's Transcendental Unity of Apperception[15] and grants *consciousness* only to beings with reflective awareness (that is awareness of being aware of bodies external to it). This reflection gives reason access to necessary truths. Leibniz is here constrained by his criterion of necessity. Leibnizian necessity is analytic—necessary propositions are those whose denials are self contradictory. Section 30 of *The Monadology* begins: "It is also by the knowledge of necessary truths and by their abstractions, that we rise to acts of reflection, which makes us think of that which calls itself 'I', and to observe that this or that is within us: and it thus implies that in thinking of ourselves, we think of being, of substance, simple or composite, of the immaterial and of God himself..." Perhaps he is suggesting that the mind concentrates on the subject because the truth value of a necessary proposition is determined without reference to an object.

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> ***...his life's goal [was] to create an 'alphabet of human thought,' the symbols of which if "correctly and ingeniously established...will be capable of being read without any dictionary" and will provide "a fundamental knowledge of all things."***

Kant employs a second test of necessity to tell a more plausible story. If something is universal, that is if it applies to everything, there is some sense in which it is necessary.[16] Kant notes that " [i]t must be possible for the 'I think' to accompany all my representations."[17] Though the proposition 'I think that X is my representation' is analytic, "it reveals the necessity of a synthesis of the manifold given in intuition."[18] Descartes never could have forseen the uses to which his *cogito ergo sum* would be put.

Leibniz calls those monads that are the souls of the reflective animals—those which are said to be rational—spirits and says that they have access to "immaterial things and truths."[19] "...As regards the rational soul, or *spirit*, there is something in it more than in the other monads, or even in simple souls. It is not only a mirror of the universe of creatures, but also an image of the Divinity. The *spirit*...imitates, in its department and in its little world, where it is permitted to exercise itself, what God does in the large world."[20] Reason is the pathway to the city of God.

In streamlining notation, Leibniz is making that pathway more accessible to his fellows. Thus, his life's goal to create an 'alphabet of human thought,' the symbols of which if "correctly and ingeniously established...will be capable of being read without any dictionary" and will provide "a fundamental knowledge of all things."[21] Leibniz did not finish this project before his death, but his work was carried on by people like Frege and Russell and Whitehead who formalized first order logic. That this is not possible was not proven until 1931 when Kurt Goedel showed that all first order systems contain statements that can neither be proven true nor shown to be false(e.g. "this sentence is false"). The 'universal calculus' is the first of the two *calculi* with which Leibniz would preoccupy himself. Another is the differential calculus for which he and Newton are remembered. That calculus is far more specific—not covering "the conceptual atoms from which absolutely all molecular concepts can be formed,"[22] but only the rates of change at an instant—that is, variation of variation of bodies in space. We set out to show that Leibniz's mathematics and philosophy are not clearly differentiated bodies of work, but are, respectively, his methods and his aims.

One approach to this task is to explore the philosophical pre-suppositions which Leibniz makes when he chooses to use the mathematical techniques which he does. Answering this question requires first a detailed examination of the problems and of Leibniz's solutions. Most of the English secondary literature on this topic seems to depend on J.M. Child's 1920[23] translation of the manuscripts "...unearthed by Dr. C.I. Gerhardt in a mass of papers belonging to Leibniz that had been preserved in the Royal Library of Hanover..."[24] Child's work contains both the daily notes which Leibniz made to himself in late 1675 while he was in the process of making his discovery and two later reflections upon that period: the first a postscript, later cancelled, to a letter written from Berlin in April of 1703 to James Bernoulli; the second, Leibniz's own version of the story to be told to posterity: *Historia et Origo Calculi Differentialis* which was "probably finished just before the death of Leibniz...in November 1716."[25] This last is of the same period as *The Monadology* and *The Principles of Nature and of Grace* (both 1714) as well as the *Metaphysical Foundations of Mathematics* (1715) which also warrants our attention.

Important to the discovery of the infinitesimal calculus was the friendship which Leibniz made with Christiaan Huygens shortly upon his arrival in Paris in 1672. Leibniz looked closely at Euclid's axiom that the whole is greater than the part[26] and determined that it was not valid when applied to the angle of contact between a circle arc and its tangent.[27] That the whole seems not to be greater than the part is the result of a long lived debate over the definition of 'angle.' The Ancients did not require that the line segments which bound an angle be straight. Euclid defines rectlineal angles as a species of angles in general.[28] There was also great uncertainty about into which of Aristotle's categories angles should be put. Euclid's view was that angles, as magnitudes, belong in the category of quantity. If angles are magnitudes it must be possible to relate the quantity of any two angles as a ratio (e.g. 45 degrees : 90 degrees = 1:2). Assume a circle centered at (0, -R) tangent to the X axis and normal to the Y axis at the origin. Construct a secant line from the origin to (R, -R). Together, with the X axis, it forms a 45 degree rectilineal angle L_R, the magnitude of which can be compared with that of the curvalineal, horn-like keratoeides angle L_C formed between the X axis and the circle arc from the origin to (R, -R). If we successively reduce the length of the secant we reduce L_R by the same ratio. We can form a series of congruent ratios: $L_R{:}L_C = L_{R1}{:}L_{C1} = L_{Rn}{:}L_{Cn}$ where the length of the secant is 1/n of the original. At the point of tangency, both angles are of zero magnitude and yet expected to stand in the same constant ratio. Euclid maintains that all rectilineal angles are greater than any curvilineal angle (III, 16).[29] Thus, the rectilineal angle is in the contradictory position of having to be both greater than and equal to the 'horn-shaped' angle. The failure of the axiom to hold for the angle of contact led Leibniz to believe, with Hobbes,[30] that it is not an axiom at all, but a provable theorem.[31] From early childhood Leibniz wanted to allow only definitions and statements of identity to be axiomatic.[32] He reduced Euclid's axiom to a syllogism containing only definition and identity:

"Whatever is equal to a part of another, is less than that other: (by the definition)
But the part is equal to a part of the whole: (i.e., to itself, by identity)
Hence the part is less than the whole. Q.E.D."[33]

Since any part 'A' is equal to itself, we can conclude that A - A = 0. And build from that:

"A - A + B - B + C - C + D - D + E - E = 0
\/ \/ \/ \/
+ L + M + N + P

If now A, B, C, D, E are supposed to be quantities that continually increase in magnitude, and the differences between successive terms are denoted by L, M, N, P, it will then follow that

A + L + M + N + P - E = 0
i. e. L + M + N + P = E - A

that is, the sums of the differences between successive terms, no matter how great their number, will be equal to the difference between the terms at the beginning and the end of the series." Leibniz went on to take the second differences—that is the differences of the differences—and the third. Once he got going this way, it was not possible to avoid discovering "that the differences of the powers of the natural numbers, when taken continuously, finally vanished."[34]

Leibniz went to Huygens with the news of his discovery of a general method for summing infinite series. Huygens was interested and suggested both a problem which he had considered earlier as a test exercise as well as some readings in the relevant literature.[35] Thus began the relationship to which Leibniz says he owes his "introduction to higher mathematics."[36] Huygens lead Leibniz to Gregoire's *Opus geometricum* which considers geometrical representations of sums of series.[37] Leibniz met other mathematicians in his travels. It was in London where he met John Pell who told him that Nicolaus Mercator had already documented the discovery of the vanishing differences of powers.[38] Leibniz's work in the summing of difference series led to the conception of the 'harmonic triangle' "in which the oblique rows are successive difference sequences, so that their sums can be easily read off from the scheme."[39] This representation made obvious the mutually inverse relation between difference sequences and sum sequences. This became quite significant when Leibniz applied a similar scheme to geometry and discovered that "the determinations of quadratures and tangents are also mutually inverse operations."[40] The area under a curve which crosses above the X axis at the origin can be approximated by inscribing rectangles and then taking the summation of their areas. If we assume the rectangles to be of width 1, the area under the curve is approximately equal to the sum of the Y values. Leibniz realized that if the rectangles became infinitely narrow, the approximations would become exact.[41] While the rectangles still have width, atop of each is a right triangle—the hypotenuse of which is a section of the circle arc. Blaise Pascal had noticed that these triangles are nearly "similar to the triangles formed by ordinate, tangent and subtangent, or ordinate, normal and sub-normal"[42] when proving "the theorem of Archimedes for measuring the surface of a sphere."[43] From this Leibniz was able to develop a general theorem for determining "the whole moment of the curve."[44] Leibniz spent the better part of two years "finding, analytically (that is, by manipulation of formulas) all sorts of relations between quadratures."[45] One such relation was that "the moments of the differences about a straight line perpendicular to the axis are equal to the complement of the sum of the terms, and the moments of the terms are equal to the complement of the sum of the sums."[46] Leibniz first expressed this relation in a notation borrowed from Bonaventura Cavalieri.[47] For example: "omn.xl = x omn.l - omn.omn.l, where l is taken to be a term of a progression, and x is the number which expresses the position or order of the l corresponding to it; or x is the ordinal number and l is the ordered thing."[48] ("'Omn.' is the abbreviation of *'omnes linae'*, 'all lines.'")[49] Utility rather than necessity was the mother of the invention of the integral notation of the calculus: "It will be useful to write ∫ for omn., so that ∫l = omn.l, or the sum of the l's. Thus.... ∫(xl) = x∫(l) - ∫∫l is equivalent to the 'omn.xl = x omn.l - omn.omn.l' of the above example.[50]

This elegant, economical notation freed Leibniz to easily develop an elaborate scheme of analytic truths. Elaborate as this scheme is and though it is "correctly and ingeniously established" it is not an 'alphabet of human thought' and it certainly does not provide "a fundamental knowledge of all things."[51] That Leibniz was able to dig as far as he did into the foundation of analytic thought is amazing. I can't help but believe in his belief that:

> [E]verything in the whole wide world proceeds mathematically, that is infallibly, so that if one had enough insight into the inner parts of things and also enough memory and understanding to take in all the circumstances and calculate them, he would be a prophet; he would see the future in the present as in a mirror.[52]

This gave him the zeal to proceed with his analysis. Though this belief relates mathematics with the world, it is a *philosophical* belief—not a mathematical one. In this sense it is not unlike the question of the ontological status of the *infinitesimal* — a philosophical question about a mathematical method. It is for this reason that I disagree with Child when he maintains that "the main ideas of [Leibniz's] philosophy are to be attributed to his mathematical work, and not *vice versa*."[53] For Leibniz, mathematics and philosophy are inexorably intertwined.

NOTES

[1] Anton, Howard; *Calculus With Analytic Geometry*; Wiley; NY; 1980.
[2] See for example J.M. Child; *The early mathematical manuscripts of Leibniz*, Open Court,1920, in which the author credits Barrow with the discovery (pp 7-8).
[3] Child, p. iii.
[4] *Monadology* [1714], *SS* 2. in *Leibniz Selections*, Weiner, P., ed. Scribners 1951.
[5] *The Principles of Nature and Grace, based on Reason* (*Principles*) [1714], *SS* 1, in Wiener.
[6] *Monadology, SS* 3.
[7] *Principles, SS* 2.
[8] *Monadology, SS* 10.
[9] *Monadology, SS* 17.
[10] *Principles, SS* 3.
[11] *Principles*, *SS* 10.
[12] *New Essays* [1704], p. 378 in Wiener.
[13] *Principles*, *SS* 3.
[14] *Principles, SS* 7.
[15] *Critique of Pure Reason*, Kemp Smith, translator, St. Martins, 1965 (B ed.), Transcendental Deduction *SS* 16.
[16] Robinson, Richard, *Necessary Propositions*, Mind (1958) pp. 291.
[17] *Critique of Pure Reason SS* 16, (B 131).
[18] *Critique*, B 135.
[19] *Principles, SS* 5.
[20] *Principles, SS* 14.
[21] *De Arte Combinatoria*, [1666] (p. 90)
[22] *Leibniz, Logical Papers*, G.H.R. Parkinson, Trans. & Ed., Clarendon, Oxford, 1966 p. XV.
[23] Child, J.M., *The early Mathematical Manuscripts of Leibniz, Translated from the Latin texts published by Carl Immanuel Gerhardt with critical and historical notes*. Open Court, 1920.
[24] Child, Introduction, p. 3.
[25] Child, p. 4.
[26] *Elements I*, Common Notion 5. (New Numbering) Axiom 8 (Old System), Heath, Sir Thomas L., Trans., Dover 1956.
[27] Hofmann, Joseph Ehrenfreid, *Leibniz in Paris 1672-1676, His Growth to Mathematical Maturity*, Cambridge, 1974, p.12.
[28] Heath, Definitions 8 & 9.
[29] Heath, p. 177.
[30] Thomas Hobbes, *De corpore* I Ch. 8 *SS* 25.
[31] Hofmann, p. 13.
[32] *Historia Et Origo*, in Child, 1920, p. 29.
[33] *Hist. Et Origo*, p. 30.
[34] *Hist. Et Origo*, p. 36.
[35] Hofmann, p. 15.
[36] *Hist. Et Origo*, p. 36.
[37] Hofmann, p. 166.
[38] *Hist. Et Origo,* p. 36.
[39] Bos, H.J.M., *Newton, Leibniz and the Leibnizian Tradition*, in I. Grattan-Guinness, ed. *From the Calculus to Set Theory, 1630-1910*, Duckworth, London, 1980, p. 61.
[40] *op. cit*. pp. 61-2.
[41] Ibid
[42] Bos, p. 63.
[43] *Hist. Et Origo* p. 38.
[44] *op. cit*. p. 40.
[45] Bos, p. 66.
[46] MSS. October 26,1675, in Child, 1920, p. 70.
[47] Bos, p. 66.
[48] MSS. October 29, 1675, p. 80.
[49] Bos, p. 66.
[50] MSS. October 29,1675, p. 80.
[51] *De Arte Combinatoria*, (90).
[52] *On Destiny or Mutual Dependence*, in Wiener, p. 571.
[53] Child, Introduction, p. 4.

"It is the theory that decides what we can observe."
--Albert Einstein

Book Announcement: *Women, Art and Geometry in Southern Africa*, by Paulus Gerdes

Women, Art and Geometry in Southern Africa. Paulus Gerdes. African World Press, 1998. ISBN: 0-86543-601-0 (hardback), 0-86543-602-9 (paperback)

Femmes et Géométrie en Afrique Australe. Paulus Gerdes. L'Harmattan, 1996.

The book was originally published in 1995 by the Universidade Pedagógica in Mozambique and received the Special Commendation in the 1996 NOMA Award for Publishing in Africa Competition. The book was praised by the jury as "combining in an ingenious way the study of geometry with that of the visual arts, presenting an important challenge and stimulant to the future of mathematics education in Africa. It demystifies mathematics in relation to gender and race, and erases the borders between mathematics and popular culture as experienced in the work and crafts of women in southern Africa. The book's importance lies in its prospective impact on the education of African women in mathematics."

African peoples in general, and those in Southern Africa in the post-apartheid era in particular, are facing the urgent need to awaken and nurture their magnificent creative potential for the benefit of all. Women, constituting half of the population, are still strongly underrepresented in scientific and technological careers where mathematical ideas play an important role.

Outside the school context, Southern African women have been involved in cultural activities--such as ceramics, beading, mural decoration, mat and basket weaving, hair braiding, tattooing, string figures—which bear a strong artistic and mathematical character.

Mathematics is the science of patterns. Southern African women have created and continue to create, invent, and imagine beautiful patterns. Some of these patterns from mat and basket weaving, ceramics, tattooing, string figures, beading, and mural decoration, are presented in the book.

The main objective of the book is to call attention to some mathematical aspects and ideas incorporated in the patterns invented by women in Southern Africa (Angola, Botswana, Lesotho, Malawi, Mozambique, Namibia, South Africa, Swaziland, Zambia, Zimbabwe). It is the author's wish to contribute to the valuing, revival and development of these traditions and their incorporation into (school) education. As an example of the educational use of female decorations, the book presents the reinvention of the Theorem of Pythagoras.

The author Paulus Gerdes is a Mozambican scientist and artist, who is professor of mathematics at Mozambique's Universidade Pedagógica. Dr. Gerdes was the Rector of this university from 1989 to 1996. From 1996 to 1998 he was visiting professor at the University of Georgia (Athens, USA). Within the African Mathematical Union, since 1986 he has been the chairman of the Commission for the History of Mathematics in Africa (AMUCHMA). He was from 1991 to 1995 the Secretary of the Southern African Mathematical Sciences Association (SAMSA). He published several books on mathematics and mathematics education in Africa, among which the following have appeared in English:

Lusona: Geometrical Recreations of Africa (latest edition by L'Harmattan, Paris, 1997)

Sipatsi: Technology, Art and Geometry in Inhambane (co-author Gildo Bulafo, Universidade Pedagógica, Maputo)

African Pythagoras: A Study in Culture and Mathematics Education (Universidade Pedagógica, Maputo)

Ethnomathematics and Education in Africa (University of-Stockholm's Institute of International Education)

Sona Geometry: Reflections on the Sand Drawing Tradition in Africa South of the Equator (Universidade Pedagógica, Maputo.