Robert Davis: In Memoriam

A recurring theme throughout Bob Davis's writings was the difficulty of using words. He wrote extremely carefully, often with poetic eloquence. As his chapter heading "Neither Words nor Pictures" indicates, mathematics seems to lie somewhat beyond the reach of images and language, no matter how precise we try to be. The same holds true, he emphasized, of most ideas worth writing about. Bob's subtlety was daunting to some readers, like his hesitation to make snap pronouncements, his discomfort lest a major issue become flattened by superficial thoughts or words. Facts and dates can never sum a person up, but still they tell us something.

Robert B. Davis was born in Fall River, Massachusetts, a son of the late Benjamin Franklin Davis and the late Ethel (Reed) Davis. He received his doctorate in mathematics at MIT in 1951. Early in his career, as a faculty member at Syracuse, Bob founded and led the Madison Project, an innovative school mathematics program, based on children's thinking and strong teacher participation. The Madison Project's materials, as well as its ways of working, have had decisive influence on the profession as a whole. Named after the inner-city school in Syracuse where the work began, the project helped to pioneer a pedagogy which emphasized careful and attentive response to how students thought about the mathematics they explored. It soon had impact nationwide, and then beyond.

The theme of helping disadvantaged children threads through Bob's whole life. So does careful thinking about learning and cognition, investigated through beautiful and seminal case studies. As we read these, even decades later, we can hear the voices of the children whom he cared so much about. As Director of the Madison Project, Davis would lead up to ten week-long seminars each summer, teaching between 100 and 1000 teachers each week in cities from New York to Chicago to San Francisco. The methods and materials developed by the project were designed, and later demonstrated to be successful, for children from all walks of life. These materials, and others based on them, are still in use throughout the world.

Bob worked at Syracuse University, at Webster College in Missouri, and at the Center for Research in Education at Cornell before moving to the University of Illinois in 1972. There he was Professor of Education, Associate Director of the computer-based Education Research Laboratory, and Director of the Curriculum Laboratory. At Illinois he continued his fundamental work on mathematical cognition, especially through a series of important studies of learners' mathematical thinking and problem solving. This theoretical orientation, based on key ideas of cognitive science, informs his later work.

He also served on the Research Advisory Cominittee for the National Council of Teachers of Mathematics, as a member of many National Science Foundation Advisory Committees, including the NSF continuing panel on the Educational Uses of Computers, as an advisor on mathematics programming to Children's Television Workshop, and as advisor to many state education departments.

In 1971, together with Herb Ginsburg, Bob founded the Journal of Children's Mathematical Behavior, which he edited until his death. The first issue described its unique focus, on what mathematical thought means with children, how it develops, and how one might attempt to study it. The second word in the Journal's title soon seemed too restrictive, and eventually was dropped. The Journal soon became, and has remained, an important sounding board for new ideas in mathematics education.

Bob's work helped to define the field of mathematics education, and his insight opened new ways of thinking about how people learn and do important mathematics. From the start, Bob's focus was on people: how they thought, what they did, how they learned from their experience, and most especially —recorded with precision, analyzed with knowing, sympathetic care — how they expressed themselves and their ideas. Bob taught, in fact, the way he did research: mainly by listening. As he listened, both he and those who spoke with him would change.

Bob's book, *Learning Mathematics: the Cognitive Science Approach*, which appeared in 1984, laid out a new agenda for research in mathematics education. This agenda, elaborated in the JMB, in part through papers of his own, in part through editorials on other people's work, continues to unfold. The resulting ferment of perspectives and ideas was heightened by Bob's move, in 1988, to Rutgers. His later work took place within what had now become a wide discussion among collaborating teachers, research scholars, and educational activists, sparked by his continued research on the development of mathematical ideas in children and adults. The communities that evolved around his work found voices, listened, and responded to each other.

Bob often reminded us that strong communities survive their founders, by finding ways to keep on growing. Now we need to demonstrate this truth by how we put it into practice. Bob resisted sentiment in place of honest passion and clear, committed thinking. He didn't want to see his life and work compressed to sound bites, chunks, or easy summaries. We miss him everyday. We miss the power of his listening. We miss his happy laugh, his deep, unflinching care for people, his graceful modesty. When he really liked something, he would say, "This is gorgeous!" and his face would shine. We miss his pleasure in the work of others.

Survivors include a son, Paul of Ann Arbor Michigan; a daughter, Alexandrea Davis-Hay of Bradenton, Florida; his wife, now Rose Garcia of Apopka, Florida; a brother, Edward of Leverett, Massachusetts; a special friend, Mary Howard of Highland Park, New Jersey; and many thousands of young people, most of whom have likely never heard Bob's name, whose lives he cared about, and touched, and changed.

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MATH POEM

Math is easy as A, B, C
It deals with all numbers
like one, two, and three

It's simple and easy and fun and it's great
I can multiply see!
Two times four equals eight

With angles and shapes and parallelograms and more
How many right angles in a square?
I know! It is four!

Denominators, numerators, fractions galore
Math is so fun, it's never a bore
With so many things to know and explore!

Beth Corridori

THE POEM OF MATH

I saw a circle in my book,
It was so round I had to look.

On the next page was a square,
Its four sharp sides gave me a scare.

A triangle fell out loose,
I do believe it was obtuse.

Now the next page was angley,
Then it was very tangley.

Then I realized it was an angle,
So I named it very very tangle.

A cylinder popped out of my book,
It was round I had to take a look.

Michelle Wang

Randal Bishop Plunges into 4-D Space

A comment on Randal Bishop's paper titled "The Use of Realistic Imagery to Represent the Relationships in a Four-Dimensional Coordinate System"

C. Ernesto S. Lindgren
Federal University of Rio de Janeiro, Brazil

I never had the pleasure of meeting Randy Bishop personally but for the last five years we have talked by telephone and exchanged mail. This contact started shortly after he presented his paper at the meeting in Australia. When I first read it - and at that time I was in Brazil - I was surprised to see that, after all, some interest had been generated by the book that I co-authored with Professor Emeritus Steve M. Slaby of Princeton University in the academic year of 1964-1965. Since Steve Slaby and I started being interested in 4-D geometry about the same time - Slaby in Norway, around 1953, and I in Brazil in 1954 - we realized that little by little our ideas would take hold and involve more and more people. I cannot say that they are, at this time, in sufficient number to fill a bus or even a mini-van. But the point is that the message that we have being trying to get across finds a new voice in Randy Bishop. He indeed plunged into 4-D space. He breathes 4-D geometry.

In this sense his enthusiasm reminds me of my own, almost half a century ago. The difference, however, is that he chose an alternative path that neither Steve Slaby nor I - nor many other people interested in 4-D geometry - had foreseen. If the late David Brisson is the sculptor of 4-D structures, Randy Bishop is the photographer. As I see it, if not Bishop then someone else will merge the two proposals: that of Brisson and the ideas outlined by Bishop in his paper. Still, why is it so difficult to generate interest in ideas that have been dealt with for more than one century?

In my conversations with Randy Bishop and Steve Slaby I have said that it is very hard to ask others to think about what has not been thought about before. This proposal, however, has to be dealt with quite differently from that which is taken by the inventor of a not as yet existing gadget, for instance. To plunge into 4-D geometry one has to start from very basic questions, much like Randy Bishop has been doing. For instance, it is prejudicial to pre-assign a type of 4-D space with which we would want to work. By this I mean that it is not import to pre-decide if that space is Euclidean or non-Euclidean. Start with what we have and then ask some pertinent questions. For instance, I dare to assume that as Randy Bishop read for the first time my and Slaby's book what he asked himself was: if all this is correct, would it be possible to photograph a 4-D object? In other words, his interest was focused on doing something that appeared feasible and that had not been thought about before. It mattered not if the Four-Dimensional Descriptive Geometry that was proposed applied to an Euclidean space. The main thing was "to think 4-D" realizing that, as a former professor of mine, Fellipe dos Santos Reis, observed, all geometric systems can be transformed into each other: a straight line can be warped and transformed into something else but always be called a line. Thus what Randy Bishop seeks to achieve is to obtain a photograph of a 4-D object. Period. But to make things easier he will be working with an object that would be familiar to us all. Let me make additional comments about this.

If we consult Henry Poincaré writings of the beginning of the century we find that he at one time made a very curious question: How do we know if we expanded or contracted in length, width and breath from one moment to another? The answer is that we cannot, for everything else will have expanded or contracted in the same proportion. Thus we cannot perceive if our 3-D space is Euclidean or non-Euclidean. It matters not. But, if it makes things easier to work, say, with a cube constructed according to the Euclidean geometry and this at an infinitesimal scale as we relate it to the Universe as a whole, then let us do it. We can always accept the fact that that cube obeys the rules of some other geometry, rules to which we are subject. We can then talk about a "line." Never mind what kind of line it is. It might not be the Euclidean straight line, but if we start with the assumption that it is Euclidean, the next question should be addressed not to its properties but to finding out how it came about.

This observation brings me to my own reasoning some 43 years ago that led me into the world of 4-D geometry. I had read somewhere, still as a young boy, that there are no straight lines in Nature. That was a very interesting thing because, after all, I was immersed in the study of the Euclidean geometry just as boys and girls of today are. Straight lines, planes, triangles, angles, and all sort of geometry forms. It appeared strange to be studying geometric forms whose basic element had no correspondence with reality. The question then was: if we cannot find a straight line in Nature, how was the first straight ruler constructed or fabricated? Later on I came upon another curious remark: to build a geometric system using only a compass. Can it be done? Well, the answer is yes. In fact, Euclidean geometry starts exactly this way. Euclid had no ruler to draw a straight line. How did he do it then? The solution is simple. We get hold of two branches of a tree and tie together one of their ends. We have a compass. With one end mark two points on the ground. With center in these points we can draw sections of circles that intersect at points. If we vary the opening of the compass we obtain a set of points. And these points, if they are infinitesimally distant, define a straight line. We can now take a piece of some pliable material and adjust it to that line, obtaining the edge of a ruler. With this edge we can replicate the straight line.

It matters not to pre-decide if the "surface" (ground) upon which we carried on the exercise is an "Euclidean plane" or if it follows the properties of some other geometric system. What matters is that now we have some means of dealing with the intrinsic geometry of a 2-D space which we can at least see. This done we can now, yes, chose a particular geometric system to analyze what goes on that "plane." It appears that I am reasoning in circles, but in fact what I am trying to say is that we must start without pre-conceived notions. I observe that with the compass I was able to generate a geometric form that can be repeated many times but now without the use of the compass. The construction of the ruler constitutes "to think about something that had not been thought before:" how to construct a geometric form without using the same instrument applied for its definition. Next we realize that that "line" can be drawn in a totally, independent "plane" and that the "lines" might not meet even though the two "planes" do so.

If we take this line of reasoning and apply it to a "cube" we must ask the question: can I replicate that "cube" in some other, totally independent "space," a "space" with the same properties as those of the "space" into which the first "cube" was initially constructed?

The answer ought to be yes. Conceptually, I mean. And then there is no other alternative but to conceptually admit that the two "cubes" (three-dimensional things) can only coexist within a four-dimensional thing.

That is what we have to hear when Randy Bishop exclaims: "we are four-dimensional beings." He is absolutely correct. A point is "at home" in a line. A line is "at home" in a plane. We all, three-dimensional beings, are "at home" in a 4-D space. If this is not so we are all, then, Abbot beings.

What we have to hope is that sometime, someone, somewhere, will achieve Randy Bishop's proposal: snap a picture as we leisurely float within 4-D space. Then we will be able to confirm if we constantly contract and expand as time goes by, in and out of "straight" 3-D spaces, through warped Riemannian spaces or in a never ending loop within a gigantic Mobius-type 3-D space.

But before this, let us have some more of Randy Bishop's news on how does it feel to plunge into 4-D space.

The Use of Realistic Imagery to Represent the Relationships in a Four-Dimensional Coordinate System

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SUMMARY
First, this paper will describe the fundamentals of dimensionality and the relationships of the elements in a four-dimensional coordinate system. Second, it will outline a number of analogies between the elements and techniques of descriptive geometry with the elements and techniques for creating images of one-point perspective. Then, it will describe, at least, two of the ways realistic imagery can be used to illustrate all of the relationships in a four-dimensional coordinate system. The use of realistic imagery will allow one to decipher the illusions inherent in diagrams of higher dimensions. This will also allow for the illustration of a four-dimensional coordinate system with the images of any chosen space.

1. INTRODUCTION

A four-dimensional coordinate system, in the terms of descriptive geometry, is composed of one, two and three dimensional elements, as will be explained and illustrated. It is now common for us to consider the space we live in and the way our eyes see objects in space to be "three-dimensional" or "3-D." These "realistic" pictures are actually illusionistic images of perspective also termed one-point perspective, three-point perspective, linear perspective, centric perspective, artificial perspective, scientific perspective and geometric perspective. The easiest way to explain and capture an image of one-point perspective is with a photograph. One-point perspective can also be recreated with any aesthetic medium: drawing, oil painting, sculpture, hologram, virtual reality or any other technique for creating a motion picture. A detailed explanation of the composition of an image in one-point perspective and an introduction to the history of perspective in art is in Section 3.

2. DESCRIPTIONS OF DIMENSIONALITY

Four-Dimensional Descriptive Geometry by C. Ernesto S. Lindgren and Steve M. Slaby is a step-by-step explanation with accompanying diagrams showing the ways the lower dimensional elements of descriptive geometry, commonly called lines, planes, and boxes, form a four-dimensional coordinate system. These are the same geometric elements in a three-dimensional Cartesian coordinate system: three mutually perpendicular one-dimensional lines forming three mutually perpendicular two-dimensional planes, all intersecting at a zero-dimensional point, to form a three-dimensional coordinate system.

Similarly, a four-dimensional coordinate system has four mutually perpendicular one-dimensional lines, six two-dimensional planes and four three-dimensional spaces intersecting at a zero-dimensional point (Figure 1, hypertetrahedron). There are other relationships and requirements for the elements of a four-dimensional coordinate system. Still, these descriptions suffice as an extremely abbreviated and simplified explanation of dimensionality and the terms higher-dimensional, multi-dimensional and n-dimensional.

These examples and descriptions do not encompass n-dimensional topological models.

3. SIMILAR CONCEPTS OF SPACE

It is also common for us to associate one, two and three dimensions with an image of one-point perspective: a one-dimensional line with the line of sight perpen-

Figure 1 (hypertetrahedron) corresponds to the diagram on page 17 in Lindgren and Slaby's book.

dicularly intersecting a projected plane in the middle ground of an image, two-dimensional surfaces with that projected plane and the surface the image is on and a three-dimensional space with the whole image. Leonardo da Vinci achieved these results relying on his studies of perspective. From those studies, he developed a compositional structure for recreating accurately scaled images of one-point perspective.

His fresco, *Last Supper*,1495-1497, is the most noted example of linear perspective because its compositional lines are blatantly apparent, and it still exists. The compositional lines appear as the edges of the picture frames, walls and ceiling beams. When visually continued, these lines converge with the line of sight at the point of infinity (marked by the figure of Jesus) to create an accurate image of one-point perspective. This line of sight is also perpendicular to the projected plane, in the middle ground, where the Apostles are arranged. It is acceptable to consider this plane, where the Apostles are arranged, separate from the general compositional structure denoted by the architecture and the figure of Jesus. This plane divides the space in front of the figures from the cavernous space of the room behind the figures. The relevant aspects of this composition are the following. There is a line of sight (a one-dimensional line) perpendicular to a two-dimensional plane that can be interpreted as dividing the three-dimensional space into two three-dimensional spaces. For more details about the composition of this work, see John Canaday's book, *What Is Art*?

This type of compositional structure naturally presents the appearance of a foreground, middle ground and background. The usage of these terms in this paper will be limited to these very simple and traditional definitions. The foreground of an image, generally the lower third of the image, shows the space the artist was standing in. Therefore, the foreground shows the space represented by the photograph. The middle ground shows where the line of sight perpendicularly intersects the projected plane at the horizon line. The background is the space behind the foreground and the projected plane in the middle ground.

Leonardo da Vinci was not the first person to study perspective during the Renaissance. Leon Battista Alberti is credited for writing the first definitive treatise on centric perspective entitled *de Pictura,* 1435. The most authoritative and cited English translation to date is by Cecil Grayson entitled *On Painting and On Sculpture.* The origin of geometric perspective in art would not be complete without mentioning Alberti's dedication in 1436 of his treatise to Filippo Brunelleschi. Filippo Brunelleschi is alleged, by his biographer Antonio Manetti, to have created the first two, now lost, paintings of scenes in perspective of Florence, Italy. The exact dates of the panels are not known; however, in Samuel Y. Edgerton Jr.'s book, *The Renaissance Rediscovery of Linear Perspective,* Edgerton dates at least one of the panels to 1425 because of other paintings by Masaccio and Masolino dating from circa 1425-1427. Edgerton's assertions are thoroughly researched and are enlightenedly reified with a photographic recreation of Brunelleschi's first painting in perspective.

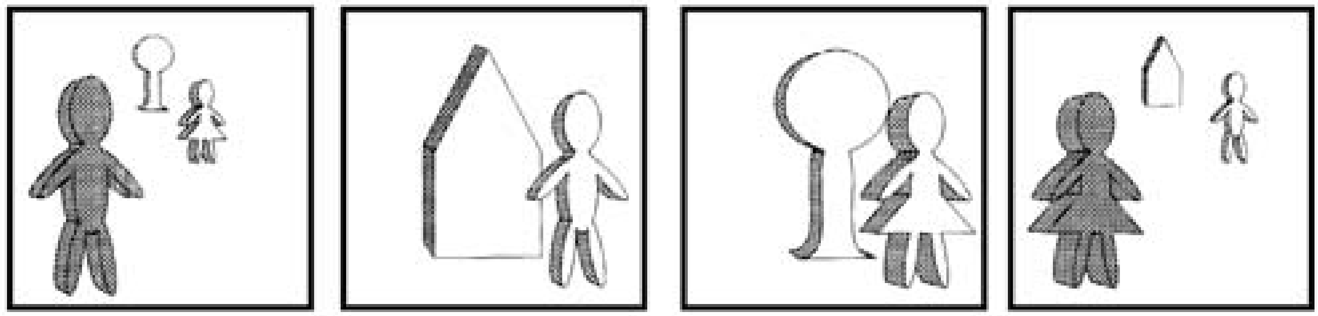
The writings of William M. Ivins, *On the Rationalization of Sight* and particularly *Art and Geometry,* link the understandings various cultures throughout the ages had of optics and geometries to the developments of linear perspective and modern geometries. Irwin Panofsky's book, *Renaissance and Renascences in Western Art,* thoroughly documents all of the aesthetic developments leading to accurately scaled, realistic images in perspective. The connectivity of art and geometric techniques through the phenomenon of perspective has been documented by many writers during the past six centuries.

In summation, if any aesthetic medium can be used to simultaneously represent the lower dimensions, then it can be used to illustrate the higher dimensions. In other words, given the assimilation of the elements of descriptive geometry with the elements of pictorial composition, any aesthetic medium can use realistic imagery to visually present the geometric relationships of the lower dimensional elements in higher four-dimensional coordinate systems.

4. GEOMETRIC RELATIONSHIPS

Lindgren and Slaby's four-dimensional hypertetrahedron, as mentioned before, is composed of four mutually perpendicular one-dimensional lines, six two-dimensional planes and four three-dimensional spaces. The mutually perpendicular requirements of the four reference lines invariably leads to the mutually perpendicular relationships of the two and three-dimensional elements. The following four conditions of perpendicularity, on page 22 in Lindgren and Slaby's book, must also be met:

Figure 2



Line 3, Plane μ, Space 1 | Line 1, Plane η, Space 2 | Line 4, Plane α, Space 3 | Line 2, Plane β, Space 4

1. The *four spaces* of the system, taken in three's, determine the four lines that are perpendicular to each other and belong to the same point.

2. *Four lines* taken in three's, determine the four spaces that are perpendicular to each other.

3. *Four lines,* taken in pairs, determine six planes, which, taken in three's, form four groups of planes belonging to the same line and mutually perpendicular.

4. Any one *line* of the reference system is perpendicular to the space determined by the other three lines in the system.

All of the resulting relationships of these conditions will be stated and illustrated by the end of the paper. However, this is not the place for a reprint of all of the geometric theorems in Lindgren and Slaby's book, some of which were originally appropriated from Henry P. Manning's book, *Geometry of Four Dimensions.* This paper will show at least one of the ways images of one-point perspective can be used to illustrate all of the relationships in a four-dimensional coordinate system.

The next step is to state a pattern for establishing four images of spaces that can be used to visually present all of the before stated elements and conditions of belonging and perpendicularity. This pattern was wholly derived from the requirements of the four reference lines. This is a logical approach for illustrating a linear coordinate system because the existence and relationships of the elements in the coordinate system are dependent upon the reference lines of the system. Then, in an analogous manner, once the four reference lines are represented by four images, the images themselves create the necessary one, two and three dimensional elements. This allows the inherent compositional structure of the images to present the one, two and three-dimensional elements in the higher four-dimensional coordinate system. The easiest way to establish four images to visually present the requirements of the four reference lines is with four photographs derived from three points in opposition. The two photographs taken from the two outer points are directed toward the central point, while the two photographs taken from the central point are directed toward the outer points.

Then each of these photographs or images must simultaneously represent a line, plane and three-dimensional space. This is achieved when the spaces are numbered, from left to right, 1, 2, 3, 4 and respectively designated as planes μ, η, α and β and lines 3, 1, 4 and 2. Planes γ and δ, the other two planes in the system, must be mutually perpendicular to planes μ, η, α and β. These relationships occur when plane γ shows the ground or horizontal plane of spaces 2 and 3, and plane δ shows the ground or horizontal plane of spaces 1 and 4. The side of viewing the spaces between the three points in opposition does not matter or if the photographs are numbered from right to left, as long as space 1 is associated with plane μ and line 3, space 2 with plane η and line 1, space 3 with plane α and line 4 and space 4 with plane β and line 2; the geometry is composed of reciprocal statements, therefore, the result will be the same four arrangements. The reciprocal nature of this geometry allows for, at least, these two aesthetic interpretations of any space chosen to illustrate a four-dimensional coordinate system.

When choosing a space, remember coordinate systems were developed for measuring space, so each photograph should obviously show the same natural space. In the natural space used to illustrate the system described in this paper, there is a house to the far left, inside of space one, a man at the plane dividing spaces 1 and 2, a woman at the plane dividing spaces 3 and 4 and a tree to the far right, inside of space 4. The front surfaces of the man, woman, house and tree are not shaded, while their sides and backs are shaded (Figure 2).

5. DESCRIPTIONS OF DIAGRAMS

Now, all of the conditions of belonging and perpendicularity of a fluctuating hypertetrahedron will be indicated with words and diagrams in lieu of real time images.

Plane η, belonging to spaces 1 and 2, is shown in clear view as a reference plane of space 2 because the foreground of the image shows the space of space 2 (Figures 3a-3e, page 10). This image also shows line 1 perpendicularly intersecting space 1 at plane η. Reference line 1 and spaces 2, 3 and 4 and planes α, β and γ become perpendicular to space 1 when reference lines 2, 3 and 4 are made perpendicular to line 1 and each other in three groups two lines, while the third line remains back to add depth to space 1. If the third line is not used to add depth to space 1, then space 1 collapses or unfolds.

Plane β, belonging to spaces 2 and 4, is shown in clear view as a reference plane of space 4 because the foreground of the image shows the space of space 4 (Figures 4a-4e, page 10). This image also shows line 2 perpendicularly intersecting space 2 at plane β. Reference line 2, spaces 1, 3 and 4 and planes α, β and μ become perpendicular to space 2 when reference lines 1, 3 and 4 are made perpendicular to line 2 and to each other in three groups of two lines, while the third line remains back to add depth to space 2. If the third line is not used to add depth to space 2, then space 2 collapses or unfolds.

Plane μ, belonging to spaces 1 and 3, is shown in clear view as a reference plane of space 1 because the foreground of the image shows the space of space 1 (Figures 5a-5e, page 11). This image also shows line 3 perpendicularly intersecting space 3 at plane μ. Reference line 3, spaces 1, 2 and 4 and planes β, δ and η become perpendicular to space 3 when reference lines 1, 2 and 4 are made perpendicular to line 3 and to each other in three groups of two lines, while the third line remains back to add depth to space 3. If the third line is not used to add depth to space 3, then space 3 collapses or unfolds.

Plane α, belonging to spaces 3 and 4, is shown in clear view as a reference plane of space 3 because the foreground of the image shows the space of space 3 (Figures 6a-6e, page 11). This image also shows line 4 perpendicularly intersecting space 4 at plane α. Reference line 4, spaces 1, 2 and 3 and planes γ, μ and η become perpendicular to space 4 when reference lines 1, 2 and 3 are made perpendicular to line 4 and to each other in three groups of two lines, while the third line remains back to add depth to space 4. If the third line is not used to add depth to space 4, then space 4 collapses or unfolds.

6. CONCLUSIONS

Realistic images of one point perspective can help us decipher the illusions in four-dimensional coordinate systems. The realistic images allow us to visually determine the "ground" planes, "side" planes and "top" planes of each of the three-dimensional spaces in a four-dimensional coordinate system. People can now visualize themselves, others, architecture or natural environments in four-dimensional coordinate systems. This art should nurture an understanding of optical illusions and Euclidean, Cartesian and non-Euclidean concepts of space. In turn, these contributions will hopefully emphasize the importance of art in the development of other disciplines.

7. ACKNOWLEDGEMENTS

The extraordinary lifetime achievements of Steve M. Slaby and C. Ernesto S. Lindgren served as a catalyst for this breakthrough in higher dimensional illustration. The author extends his deepest thanks for their unreserved, enthusiastic support. Additional thanks to Arthur L. Loeb, at Harvard University, for his thoughtful contributions. Finally, the superb graphic illustrations have evolved to their current calculated clarity with the advice and tenacious computer graphic imaging, typesetting and editing expertise of George D. Beelen Jr. of Youngstown, Ohio.

8. BIBLIOGRAPHY

Alberti, Leon Battista, *On Painting and On Sculpture*, edited with translations, introduction and notes by Cecil Grayson, London: Phaidon,1972.

Canaday, John Edwin, *What is Art,* New York: Alfred A. Knopf, Inc.,1980.

Edgerton, Samuel Y. Jr., *The Renaissance Rediscovery of Linear Perspective,* NewYork: Basic Books, Inc., 1975.

Ivins, William Mills, *Art and Geometry,* Cambridge: Harvard University Press, 1946.

Ivins, William Mills, *On the Rationalization of Sight,* New York: Metropolitan Museum of Art, 1938.

Lindgren, C. Ernesto S. and Slaby, Steve M., *Four Dimensional Descriptive Geometry,* New York: McGraw-Hill, 1968.

Manning, Henry P., *Geometry of Four Dimensions,* New York: Dover Publications, Inc.,1956.

Panofsky, Erwin, *Renaissance and Renascences in Western Art,* Copenhagen: Russak, 1960.

International Society for the Interdisciplinary Study of Symmetry (ISIS-Symmetry)

The Fourth International Congress and Exhibition of ISIS-Symmetry

TECHNION I.I.T. – ISRAEL INSTITUTE OF TECHNOLOGY, HAIFA 13 - 19 SEPTEMBER 1998

ORDER / DISORDER

Organization and Hierarchy in Science, Technology, Art, Design, and the Humanities

The Fourth ISIS-Symmetry Congress and Exhibition will take place in Israel, in September 1998. The central topic of the conference will be: ORDER / DISORDER with an emphasis on the phenomenon of morphological ORGANIZATION and HIERARCHY. It is intended to continue the decade-long activity and dialogue between those concerned and interested in the subjects of symmetry and order.

These insights bridge across interdisciplinary borders, discover interconnections and common denominators between phenomena and processes and try to create a common ground for dialogue which the forthcoming Congress and Exhibition will try to ancourage and facilitate.

The program will include plenary session lectures and presentations by a selected group of scientists, scholars, and artists including short paper presentations, colloquia and workshops on conference-related topics, and the M. C. Escher centenary session. It will also feature voluntary subject teams, exhibition of original works of art, and social and cultural interaction.

For more information about the Congress, registration fees and accomodations, please visit the ISIS website at *http://www.technion.ac.il/òisis4*

Thinking about the Preparation of Teachers of Elementary School Mathematics

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RECOGNIZING THE NEED

Numerous reports and studies published over the last decade have evidenced growing concern about the preparation of persons who teach mathematics in our nation's schools. Since the quality of school mathematics is contingent upon the quality of undergraduate programs, college and university mathematics departments are being exhorted to re-examine their own programs as well as their relationships with teacher preparation programs. Referring to undergraduate mathematics programs as "flawed models" the National Research Council states:

> Unfortunately, few university mathematics departments maintain meaningful links with mathematics in school or with the mathematical preparation of school teachers . . . Only when college faculty begin to recognize by deed as well as word that preparing school teachers is of vital national importance can we expect to see significant improvement in the continuity of learning between school and college (*Moving Beyond Myths*, 1991, p. 28).

A Carnegie survey of college faculty clearly reveals this perceived lack of continuity (National Science Board, 1996, p. 1-26). Faculty representing ten countries around the world agreed that pre-college students do not receive adequate preparation in mathematics and quantitative reasoning. The U. S. faculty ranked lowest in this perception with only 15% believing that students were adequately prepared for collegiate mathematics.

Unfortunately, mathematics faculty often fail to make the connection between the perceived lack of preparedness of pre-college students and their own undergraduate programs. High school mathematics teachers are the products of these programs. They not only teach what they learned to their students but also how they learned it. The Mathematical Association of America (1991) calls for colleges and universities to seriously rethink undergraduate mathematics instruction:

> . . . the teaching of collegiate mathematics must change to enable learners to grapple with the development of their own mathematical knowledge. As we rethink the collegiate curriculum in mathematics, we must be open to new ways of presenting mathematical ideas. The standard curriculum in place for the past several decades should give way to a curriculum that weaves mathematical strands together to create new courses and new approaches to the development of ideas (*A Call for Change*, p. 39).

SEIZING THE OPPORTUNITY

In the fall of 1997, several of us in mathematics and in teacher education at Mississippi University for Women began working in close concert to respond to the need for improved preparation of teachers of school mathematics. Our teacher education program requires students who are planning to teach in grades K-8 to take two 18 semester hour concentrations in content area courses. Very few students choose a focus in mathematics. We wanted to know why mathematics was not chosen and what we could do to promote that choice. This effort at understanding was funded by a small grant from the Exxon Education Foundation.

Our learning strategy consisted of a series of small group luncheon discussions with a dozen elementary education volunteers from the undergraduate math methods course. All of these students had taken at least 9 semester hours of undergraduate mathematics; only one was completing the 18 semester hour concentration in mathematics. All of the students were women and at least half were non-traditional older students. The discussions were purposely unstructured with the only parameter a persistent probing to understand the students' perceptions of mathematics and lack of enthusiasm for its pursuit.

THE EMERGENCE OF THEMES

As the students described their high school and collegiate experiences with mathematics, several disturbing themes began to emerge. The issue of continuity between pre-college and collegiate mathematics was particularly manifest in the attitudes of the students towards college algebra. Mississippi undergraduates are required to take a course in college algebra or above where "above" refers to a mathematics course that utilizes college algebra. All of the students had opted to take the college algebra course.

Those students who had a good high school background in Algebra I and II described themselves as inadequately challenged by college algebra because of the redundancy. One student, who had taken mathematics courses through calculus in high school, said that the college algebra course made her lose interest in mathematics and opted not to take courses beyond the basic math requirements. Those students who did not have a good high school background in Algebra I and II described themselves as overly challenged in college algebra and lost confidence. They experienced college algebra as a repeat of the frustration and sense of incompetence suffered in high school mathematics.

...it became evident that these students did not experience the problem of relevance as much in other subject areas such as humanities and social sciences.

For these twelve students their initial experience of collegiate mathematics in college algebra did not evidence a lack of continuity between high school and college, but just the reverse. The problem for these students was that their college algebra experience mirrored closely their high school algebra experience. A distinctly different initial experience of college mathematics was needed. If college algebra is deemed a mathematically significant experience for undergraduates, then the challenge is to present it in such a way that the significance is both visible and accessible to the students. Fortunately, mathematicians are revisiting college algebra. New approaches are emerging which emphasize the variety of applications in which algebra can be found and which make connections between the graphical and algebraic representations of functions. These approaches go beyond the traditional drill type problems to encourage the deeper conceptual questions that make mathematics meaningful.

The next theme to emerge in our conversations related to relevance. Initially the students talked about not seeing the relevance of college math requirements for teaching elementary school math. With continued conversation and probing, it became evident that these students did not experience the problem of relevance as much in other subject areas such as humanities and social sciences. While the meaningfulness of Beowulf and Chaucer for elementary teaching are no more evident than that of college mathematics, the students did not think of it in this way. In other disciplines they seemed to be more conscious of the connections within the disciplines and the connection between themselves and the disciplines. Learning seemed more natural to them in other disciplines. As one student expressed it: "Anyone can go into English and you know you can learn it; but the same is not true of math."

In elaborating on the absence of "connections" in and with mathematics, the students talked about the rigidity of mathematics. Unlike other subjects, mathematics did not seem to lend itself to interpretation. Students not only perceived a singular "answer" to mathematical questions but also associated fixed processes with mathematical solutions. They did not see that mathematics could draw upon the multiple modes of human thought which other subject areas draw upon.

This pattern of the perceived irrelevance of mathematics in the intellectual lives of those who will be teaching it may well be at the root of why significant reform in school mathematics has been so difficult to achieve. The students with whom we talked were not adverse to learning mathematics. Quite the opposite, they wanted to understand mathematics for themselves as well as for the sake of the children they would teach. They were frustrated that the world of mathematics felt so inaccessible and so foreign. With the exception of the mathematics courses designed specifically for prospective teachers, these students did not feel that their college mathematics courses, or for that matter their high school courses, invited them into the world of mathematics. Rather, they felt like strangers in a strange land.

It is no wonder then that these students were inclined to view mathematics as something which "you have or you don't." This was another theme that we noted. Math tended not to be seen as a learned experience but rather as a genetic endowment. This view of mathematics as an inherited aptitude seemed to coincide with the students' perception of mathematics as rigidly rule-bound. The algorithmic nature of mathematics was not seen as accessible to interpretation but only to memorization. Paralleling their perception of mathematics as an endowed aptitude was their belief that mathematics was not conducive to collaborative learning with peers. They felt that the "one right answer" nature of mathematics made cooperative learning feel like cheating. They reported a sense of isolation in their mathematics classes which seemed to emanate from the absence of multiple perspectives.

...mathematics courses need to help students learn their way around and feel at home in an environment of mathematical ideas ...

MATHEMATICS AS A HUMAN ENDEAVOR

The themes which emerged from our conversations with students suggest that these students do not view mathematics as a human endeavor. It is not seen as a peopled undertaking to which their particular persons can gain access and to which they can contribute. In the words of the MAA, their experiences with collegiate mathematics have not enabled them "to grapple with the development of their own mathematical knowledge."

In an invited address on mathematical knowledgeability given at the American Educational Research Association, Carl Bereiter (1997) argued that:

> Mathematics education ought to be making students feel at home in an environment of mathematical ideas. They ought to feel at home because they have approached those ideas from different directions, used them for different purposes, raised and investigated questions about them, discovered that they can reconstruct them from parts or even perhaps create new ones. (p. 7)

His exhortation that mathematics courses need to help students learn their way around and feel at home in an environment of mathematical ideas requires attention to what Bereiter calls "the hidden substrate of what we normally recognize as mathematical knowledge." This substrate, he maintains, "is hidden because it is psychological, a property of individual minds." (p. 6) If we are to create collegiate courses which help students to make sense of mathematical ideas, then we must attend not only to knowledge of mathematics but also to knowledge of how we learn mathematics.

> At least one of the barriers to reform is a limiting psychology that does not allow educators to conceive of mathematical ideas as real things. Until they are able to do so, they will continue to turn out students who do not have an inkling that the world of mathematical ideas exists, let alone that it is a world they could enjoy working and learning in. That is why I think it behooves mathematics educators to delve deeper into theories of mind and cognition. (Bereiter, p. 8)

MATHEMATICS AS A FEMININE ENDEAVOR

About the same time that we were having discussions with the undergraduate education majors about their mathematical experiences, one of our graduate students was conducting a study in which she surveyed over 150 gifted high school students to examine the effects of gender on career and college choices. Participants consisted of rising high school juniors and seniors selected to attend a three week Governor's School program. To gain admission into the program these students were required to meet stringent academic standards.

Only 19% of the girls surveyed identified mathematics as a favorite subject. Twice as many girls cited English and History as favorite subjects. The survey consisted of questions related to motivating factors for choice of colleges and careers. Because of the open-ended nature of the questions the graduate student was able to ascertain attitudes toward school subjects and college majors. In her conclusions she noted that:

"Across the survey, students responded that males are better in mathematics and females are better in English. When questioned about a sibling's best subject, brothers were seen as more proficient in math while sisters were more talented in English."

When juxtaposed with the results of our conversation with the education majors, the survey results are provocative. While they reveal nothing new, they do elicit some surprise at the persistence of the old. Bereiter's challenge to mathematics educators "to delve deeper into theories of mind and cognition" in order to unveil the psychological substrate of mathematical knowledge may need to encompass the differences between the masculine and feminine psyches. It is generally agreed that a masculine bias surrounds mathematical ideas, but perhaps the bias actually invades the ideas. Mathematics is, after all, a human construction, and, as such, carries the characteristics of its makers. Most of the makers have been men. As more women mathematicians become makers of mathematical ideas, there may naturally evolve a mathematics that is more appealing to women.

This short essay does not purport to provide answers but simply attempts to make visible some of the complexities involved in rethinking the collegiate curriculum in mathematics. Attention to these complexities may help us circumvent the fate of much educational reform where solutions have oftentimes introduced difficulties more challenging than the original problems. Educational problems are particularly perplexing because of the incestuous nature of our profession. Persons who have been successful students in educational settings tend to reenter those settings as teachers and are inclined to perpetuate the conditions which made them successful. It may be the lack of inclination of women to pursue mathematics and the difficulty with which they do so that constitutes our best option for understanding what needs to change in collegiate mathematics.

REFERENCES

Bereiter, Carl (1997). Mathematical Knowledgeability. Invited address, SIG/Research in Mathematics Education, *American Educational Research Association*.

Mathematical Association of America (1991). A call for change*: Recommendations for the mathematical preparation of teachers of mathematics*, Washington, D. C.

National Research Council (1991*). Moving beyond myths: Revitalizing undergraduate mathematics*, Washington, D. C.

National Science Board (1996). *Science and engineering indicators*, Washington, D. C.

GEOMETRY IN NATURE

I had never noticed all the geometry;
angles and shapes in nature for all to see.
The obtuseness of a mountain peak;
the angle of a ballerina's leap.
A rainbow is an arc of colors in the sky;
repeating flowers in a collinear line, oh my.
A hummingbird in mid-air, flying free;
the vertex of its beak pointing at me.
The world is filled with geometry;
Open your eyes, really look and see.

Rachel Finkelstein

TRIANGLE

A
triangle
is the sturdiest
shape of all. they
use it to brace a ceiling
and use it to hold up a wall.
It will not bend. It is very stable.
In fact it is holding up this table. A
triangle is every builder's friend and now
my poem is at an end.

Ian Ross

Platonism and All That...

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Humanists, I take it, value people, and are willing to weigh their suggestions and beliefs, giving some potential credence to any, as yet unfamiliar, belief, just from the fact that some people hold to it.

So, as a humanist, I may consider Platonism, formalism and constructivism, all views of mathematics which are held by thoughtful and scholarly people. My mathematical instinct to mark these views correct or mistaken should perhaps be held in check, since I am looking not at mathematical propositions, but at perspectives, views and beliefs. These perspectives may be valid at different times of the day!

My day begins with teaching. In the classroom there are pupils and myself. Here I am a constructivist. My pupils have come to learn mathematics. I am here to speak, to question, to write and act in such a way that as much mathematical thinking and knowing goes on as possible. I may exaggerate, tell jokes, deviate from a deductive sequence, perform a drama or an experiment, whatever, so long as these contribute to the pupils' construction of mathematics. Whatever mistakes they make afterwards are in some degree my responsibility, reflecting the emphasis or lack of emphasis I have generated.

When the lesson is over I return to my office and try reading about the history of the mathematics I have just taught. I would like to be a constuctivist in this mode too, but the secondary history texts tend to focus on the achievements of past mathematicians, not upon the process by which those achievements were attained, and I am regularly frustrated.

I turn then to my research on when a binary operation has the left inverse property that $ab = 1 \Rightarrow a(bc) = c$, a weak form of the associative law. I look through a catalogue of loops of order 6. Very few of these objects relate to other parts of mathematics. For the most part I seem to be indulging in a "meaningless game played with meaningless marks on paper" (Hilbert) and confirming the formalist stance. The theorems that emerge *may* become useful at some time in the future, but the satisfaction I have now does not depend on that. But I am a geometer and owe so much to Hilbert. His replacement of "points, lines and planes" by "tables, chairs and beer mats" was the psychological device that let him identify unstated assumptions in our geometrical language and produce his *Foundations of Geometry* (1899). Formalism can be both useful and functional.

Now I am a grandparent, and watch grandchildren acquiring language. I see the beginnings of counting not with the comparison of sets, but with the recitation of an almost senseless rhyme:

One, two, three four five,
Once I caught a fish alive.
Six, seven, eight nine ten,
Then I let it go again.

Just getting used to this sequence of sounds provides the equipment for future mathematics. Formalism seems to be part of the story from a very early age!

I wonder sometimes what is going on when I subtract, say, two 4-digit numbers. There is a routine, which has become second nature, a process which I apply. If the process is to be effective it is best not to relapse into thinking what it means.

After the research interval I go to the coffee room and meet non-mathematical colleagues. The post-modernists are curious about mathematics. I tell them it is fashionable to knock Platonism these days, and they respond by asking whether there are historically or anthropologically inconsistent versions of mathematics available. I tell them that each of the versions of mathematics I am aware of can be understood as containing or being contained in some other version, and they return to their work having convinced themselves that the subject is absolute. Then I reflect back to my

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Real Data, Real Math, All Classes, No Kidding

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Unlike most university professors, I took the scenic route. For three years I taught mathematics and choral music at a junior high school in East LA; for six years, at the high school next door to it; and for six more years, at the community college a few miles down the freeway from both of them. So, when I work with colleagues who teach at the elementary, middle school, or high school level, I really see us all as just that: *colleagues*. I know lots of people that would be fabulous at my university doing my job, and, frankly, from time to time I wouldn't mind being back at Sierra Vista Junior High.

WHEN AM I EVER GONNA USE THIS STUFF?

So when I get the opportunity to do mathematics with colleagues around the country as part of The National Faculty Institutes and Academic Sessions, I usually try to look at myself from the other end of the pencil. What would I want if someone were going to help me in my mathematics class? What would I not want? Why?

It's clear to me what I would *not* want: to be given the same old stuff in the same old way. I wouldn't even want the same old stuff in a new way (Uri Treisman once remarked that a good pedagogy will not fix a bad curriculum). What I would want is an experience that will cause me to think differently about what I do and why I do it. That means that I have had a mathematical experience that has, I hope, taken my breath away, and, at least, has caused me (as Arsenio Hall used to say) to go "hmmmm." Consequently, that's what I try to provide for my colleagues during these Institutes: a mathematical moment that will affect their professional lives, not just because the dynamic of the shared experience, but *because of the mathematics itself*.

So what is my greatest need as a teacher? To have a good answer to the shop-worn question,

When am I ever gonna use this stuff?

I am embarassed to admit that until recently I basically ignored this question. I gave shop-worn answers like, "Math teaches you how to think, so it doesn't matter," or "Well, we can solve word problems with math," neither of which I believed in. The truth is, I didn't know. After more than two decades of studying mathematics pretty seriously, I didn't have practical applications of percents, algebra, and calculus other than the examples in the books, most of which were either contrived or trivialized, like a 23-minute sitcom that states, develops, and resolves a major crisis. I'd like to think that I taught well, and that most of my students did well, and that some of them enjoyed me and even the math itself. But I still lacked real application of the subject.

In the past, the problem was always access: how do you get hold of real data? Where do you go to look in the library for AIDS reports, carbon dioxide levels, mortality rates, or earthquake incidences? How current is printed information? And who has the time? But with the internet, all of these things are literally a moment away. And, with private companies making internet access increasingly common for educators and non-educators alike, "surfing the net" is no longer done only by computer geeks. Now almost any geek can do it.

So the search for using real data sets in my own classes has launched a somewhat second career for me. This semester I am teaching a course entitled "Calculus for Biologists," a one-semester course built around real data sets that utilize the powerful ideas of calculus, that is, how quantities change in relation to one another, in contexts that are relevant for scientists because of their reality. After only a few weeks it is clear to me how the mechanics of the language mathematics, such as algebraic manipulations, serve primarily to enhance one's understanding of and ability to describe the dynamics of a physical phenomenon. And, if the algebra is minimized, most of the ideas would be accessible to persons with considerably less mathematics background. Indeed, exposure to these phenomena would create for the student a context in which the algebraic structure could take on physical charactersitics, as was described by the Greeks so long ago, but since lost in 20th century textbooks.

What follows is an account of two such lessons that I did with two different groups of middle school teachers. The first lesson, *Earthquakes!*, was done with a group of math/science teachers in Long Beach in southern California. The second lesson, *Math from the Crypt*, was done with a group of teachers from the Mississippi Delta region. Each uses technology in a meaningful way, but is not technology-dependent. After the data are in hand, then the mathematics really begins.

EARTHQUAKES!

Most of us who live on the west coast have experienced an earthquake (some of my out-of-state colleagues say that they would *never* live in California for that reason). Indeed, southern California natives (such as myself) have lived through some memorable shakers: the Magic Mountain earthquake of 1971 (magnitude 7.0); the Whittier Narrows ("Shake and Bake") earthquake of 1986 (magnitude 6.8); and, more recently, the Northridge earthquake in January of 1994 (magnitude 6.9). Although most of us in southern California like to think we're pretty savvy about earthquakes (we know the lingo - Richter scale, epicenter, aftershock), my observation is that we actually harbor many false ideas about earthquakes. For example, how are earthquakes caused? Are they triggered by hot weather? Is "The Big One" likely to happen? Although these questions are geological in nature and require some understanding of the earth's formation, some mathematical observations about earthquake frequencies (how often) and magnitudes (how big) can provide insight. But first, try the Earthquake "Quiz" below. Most of the Californians got fewer than two out of the five questions correct.

What Do You Believe About West Coast Earthquakes?

1. Given that a "felt" earthquake (4.0 or higher) has occurred on the west coast, the chance that it is a "severe" earthquake (6.0 or higher) is about:
a) 50 % b) 25 % c) 10 % d) 5 %
e) 2 % f) less than 1 %

2. Generally speaking, earthquakes with deeper epicenters (10 km or more) will tend to be <u>more/less/about the same</u> in magnitude as those nearer the surface.

3. Southern California tends to get <u>more/less/about the same</u> number of "felt" earthquakes (4.0 or higher) than does northern California.

4. Earthquakes in southern California tend to be <u>more/less/about the same</u> in magnitude as those in northern California.

5. Small earthquakes (between 2.5 and 4.0) constitute about what percentage of all west coast earthquakes?
a) more than 99 % b) about 90 % c) about 75 %
d) about 50 % e) about 25 % f) about 10 %
g) less than 5 %

To explore answers to these (and other) questions, we turned to the internet, hunting earthquake data. While there are several good geological sites that post recent data, we found the Earthquake Laboratory at the University of Washington to be very current and easy to use. The Internet address is:
http://www.iris.washington.edu/FORMS/event.search.form.html
We downloaded earthquake data over a six-year period for earthquakes whose epicenters were in the latitude and longitude range for the west coast (from Baja to Washington). The download yielded 90 (electronic) pages worth of data, a sample of 782 earthquakes of magnitude 4.0 or greater! We used a spreadsheet program to generate the descriptive statistics for this sample (Table 1).

Table 1 yields answers to some of the "quiz" questions almost immediately. For example, the typical earthquake in northern California has a magnitude of about 4.5, same as that in southern California. However, there were about twice as many "felt" earthquakes in southern California (213 compared to 123), while northern California quakes tended to have much deeper epicenters (11.7 km compared to 6.1 km). While these answers trigger more questions that are geologic in nature (e.g., why are northern CA quakes deeper?), they do help bring one's beliefs about earthquake behavior into line with reality. Perhaps the most interesting question centers around the relative frequency of big vs. small earthquakes (Earthquake Quiz, questions 1 and 5). A graph of the number of earth-

Table 1: Summary of Earthquake Data

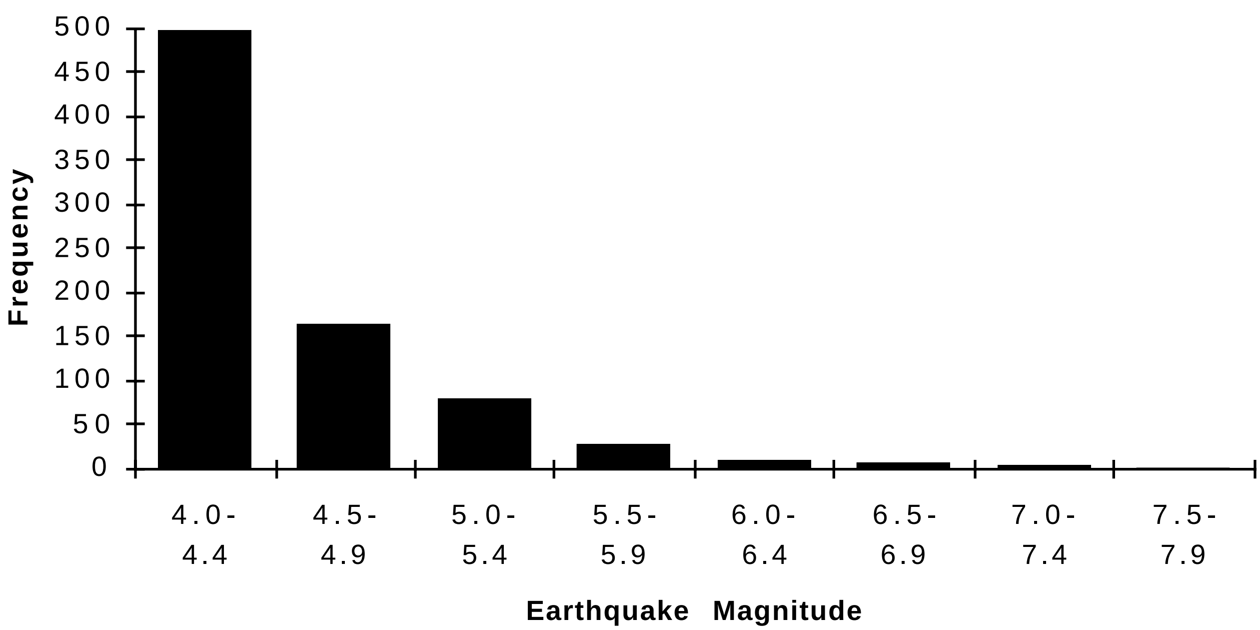
Earthquake Data by Magnitude

Line Nos.	Magnitude	n	Mean	St. Dev.	Med.	Mode	Avg. Depth (km)
2 to 2	7.5 - 7.9	1	7.6	0	7.6	7.6	1
3 to 5	7.0 - 7.4	3	7.06	0.06	7.1	7.1	13
6 to 11	6.5 - 6.9	6	6.73	0.12	6.75	6.8	14.3
12 to 20	6.0 - 6.4	9	6.16	0.13	6.2	6.3	10.6
21 to 46	5.5 - 5.9	26	5.6	0.13	5.6	5.5	8.1
47 to 124	5.0 - 5.4	79	5.18	0.14	5.2	5	8.4
125 to 286	4.5 - 4.9	162	4.67	0.15	4.6	4.5	8
287 to 783	4.0 - 4.4	497	4.15	0.14	4.1	4	7.6
2 to 783	Total Data	782	4.47	0.54	4.3	4	7.9

Earthquake Data by Region

Line Nos.	Location	n	Mean	St. Dev.	Med.	Mode	Avg. Depth (km)
1 to 17	Baja	17	4.3	0.31	4.2	4	10.3
18 to 230	Southern Cal.	213	4.49	0.55	4.3	4	6.1
231 to 331	Central Cal.	101	4.41	0.45	4.2	4	6.6
332 to 454	Cal./Nev. Border	123	4.48	0.51	4.3	4	4.8
455 to 582	Northern Cal.	128	4.49	0.67	4.2	4.2	11.7
583 to 729	Oregon	147	4.44	0.52	4	4	9.5
729 to 782	Wash./Vanc. Is.	53	4.58	0.51	4.1	4.1	10.4
1 to 782	TOTAL DATA	782	4.47	0.54	4.3	4	7.9

Distribution of "Felt" West Coast Earthquakes
January 1, 1990 through July 11, 1996 (N = 782)



quakes of different magnitude ranges (7.5-7.9, 7.0-7.4, etc.) shows an intuitive yet stunning example of exponential decline.

The data pictured in the chart above suggest that the total number T of earthquakes of magnitude M is related in an exponentially decreasing fashion. In the language of algebra,

$$T = Ab^M,$$

for some constant A and base b. Although earthquake magnitudes are measured in base ten, we found that using the base 7.1 gives a better "fit" to the data and therefore might be a better model for west coast earthquake frequencies. The data clearly show that big earthquakes (6.0 or larger) represent a fraction (18/782, or about 2.3 %) of all "felt" earthquakes over the past six years. Thus, while "The Big One" is possible, it is unlikely, and certainly does not merit the hysteria portrayed by the media every time an earthquake occurs.

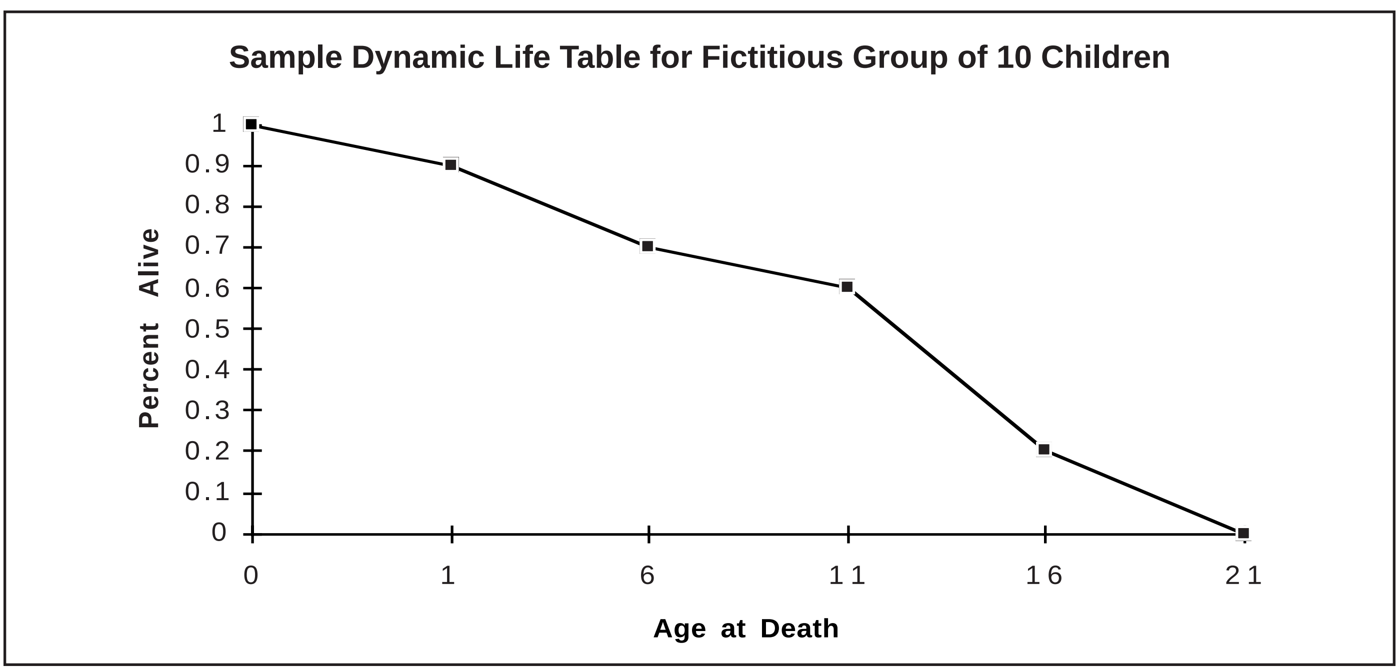
The earthquake data are a scientifically documented, accessible source of information that gives insight into understanding earthquake issues. In exploring this data, we dealt with many aspects of arithmetic and advanced algebra as well as incorporating technology in an integral way. The mathematics was powerful because it occurred in the context of a situation that was tied to the teachers' individual and collective experience.

MATH FROM THE CRYPT: INVESTIGATING THE PAST AT ST. PETER'S CEMETERY IN OXFORD, MISSISSIPPI

Description of Project

Demography is the study of the age structure and growth rate of populations. The *life table* is one way of summarizing key demographic variables, including age-specific mortality, survivorship, and expectation of further life. Once these data are compiled, we can use them to investigate demographic patterns and processes, such as differences in the survival rate or life expectancy of different groups of organisms.

The simplest way to construct a life table is to follow a group (or *cohort*) of organisms from birth, recording the age at which each individual dies, until all individuals of the original cohort have died. The result of this approach is termed a *dynamic life table*. However, cohort data are difficult and time-consuming to obtain, because the table cannot be completed until the entire cohort has died - which could take decades, in the cases of elephants or seabirds, or even centuries, as for trees such as bristlecone pines (which may live 2,500 years). Consequently, ecologists often construct life tables using other types of information. The approach we used was to gather data on the age of death of a sample of individuals, and to use these data to estimate mortality rates and to calculate other vital statistics. This approach yields a static life table, with entries that are age-specific, even though the sample is a composite, made up of individuals who started life at different times.



In this project we focused on human demography, in part because of our obvious interest in understanding the patterns and causes of death among people, but also because the data are readily available - thanks to our cultural tradition of memorializing our deceased relatives, and information about their lives, on gravestones and tombs. During our visit to the cemetery participants recorded information from gravestones. Data were separated by century of birth, sex, race, or other variables, depending on the question(s) on which teams wanted to focus. Questions which teams were addressing included the following:

•What is the general shape of the survivorship curve for your various datasets?

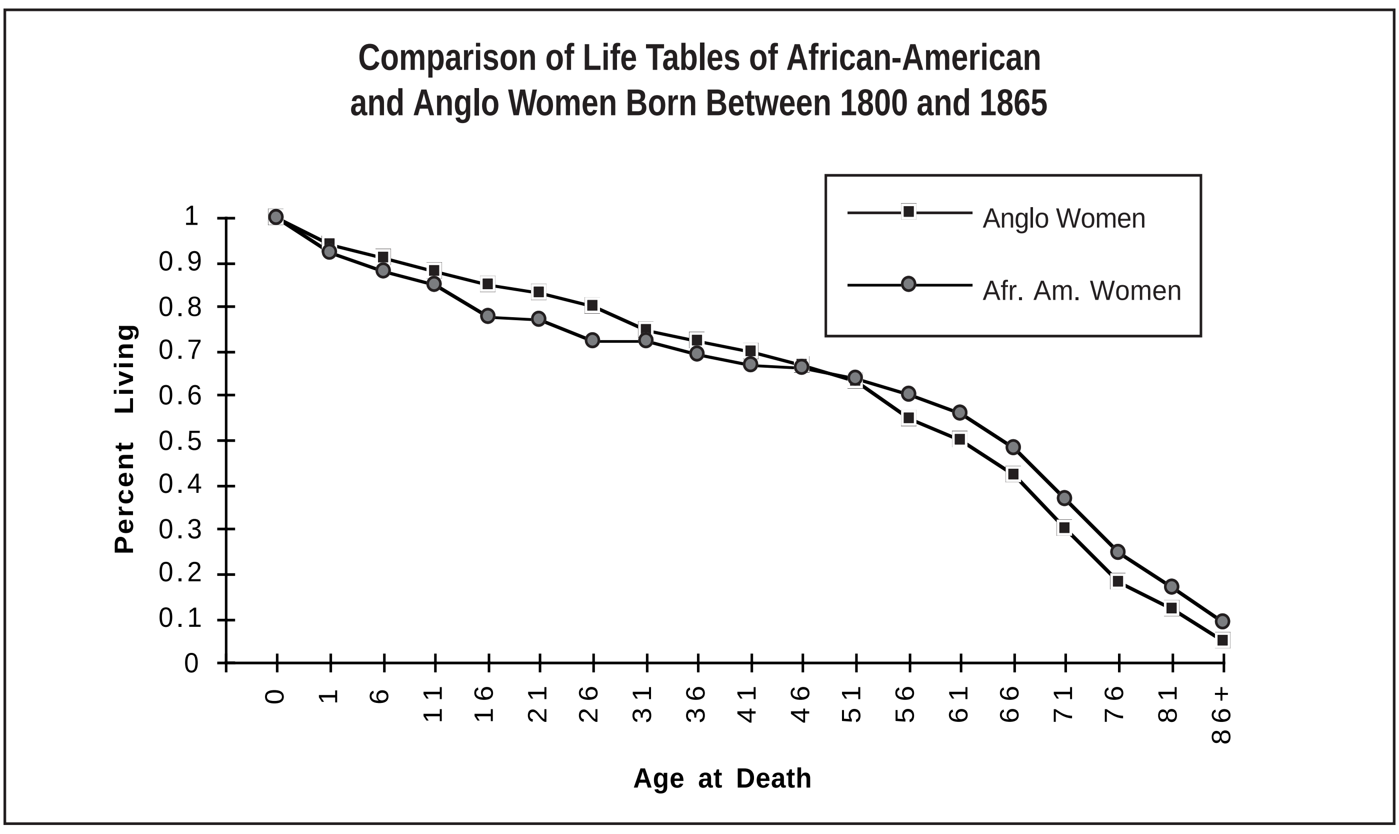
•Is there any major differences between the survivorship curves or life expectancy for people born before 1800 vs. those born in the 1800s vs. those born in this century? What biological (including medical) changes might account for any differences?

•Do the survivorship curves or life expectancies for men differ from those for women? Are the differences, if any, consistent from one century to the next? What biological factors might account for any differences?

•Do any of the datasets show marked differences compared to the recent life table for the United States population? What are the most obvious differences, and how might you explain them?

•The answers to the preceding questions might be erroneous if our data did not accurately represent the demography of people in any of the time periods. What types of biases, if any, can you envision and how might they skew the results (as well as affect our responses to the questions)?

The technique of creating life tables is a straightforward application of cumulative percentages. After recording birth and death dates, sex (inferred), and ethnicity (when known) for a sample, a simple tally is taken. Using the standard interval groupings of five years (after the first year of life), a small sample of ten children deceased before age 21 might show this: one person died at birth, two made it past their first birthday but not to 6, one person lived past six but died before age 11, four lived past eleven but died before 16, and two lived past sixteen but died before age 21. Thus, at age one, 9 people (90 %) were still alive; at age six, 70 % were alive; at age eleven, 60 % were alive; at age sixteen, 20 % were alive, and by age 21, 0 %



were alive. Thus, a graph showing survivorship rates for each age interval using this fictitious sample would be represented as shown on the previous page.

Real Data: An African Legacy of Strength

Since St. Peter's cemetery dates back to the 18th century, there were a number of comparisons that teams could make, such as comparing life tables by sex, by century, and even by race (it turned out that Section 5 of the nine-section cemetery was a pre-civil war slave section). Although some of the markers were unreadable or even absent, many of those could be identified by taking a rubbing.

Independent samples showed two findings that were consistent. First, both African-American and Anglo women outlived their male counterparts during both the 19th and 20th centuries, a result that is consistent with current lifetables for all races. Second, and perhaps less intuitive, African-American women outlived Anglo women across during both centuries, including during the age of American slavery. The graph below shows data collected by a team of African-American teachers illustrating the life tables of black women and white women born between 1800-1865. Note that the African-American group shows a marked decrease during the late teens and early twenties (probably attributed to childbirth issues), but shows a strong survivorship after age 50 and on into old age. The result depicted in this graph was corroborated by two other groups using independent samples, indicating the validity of this finding, namely, that African-American women showed stronger survivorship than their Anglo counterparts despite their status as slaves. The group whose data are shown here presented their results entirely on the computer, using overhead graphics. The group was impressive not only in its use of technology, but in its understanding of what it had found. The work of these teachers commanded the respect and admiration of all of their colleagues in the institute.

Both student and teacher are partners in trying to figure out what the data mean, and which, if any, mathematical models might be useful tools to make predictions.

SUMMARY

Using real data as a catalyst to explore mathematics has made a lot of sense to me. Both student and teacher are partners in trying to figure out what the data mean, and which, if any, mathematical models might be useful tools to make predictions. Of course, the models are far from perfect; indeed, part of the problem that scientists face is to decide which type of equation is appropriate, and over what interval is it valid. Students will disagree on solutions as well, causing a certain level of *angst* for both them and the teacher in regards to grading.

Nonetheless, I have tried to incorporate such problems into my calculus class this semester. It would be untrue to say that there have not been drawbacks. First, it takes a lot more preparation time for me to find the data sets and incorporate them in a useful and appropriate way. Second, I have less control over what the students actually learn from these types of problems, since there is often no clear answer (or even question, for that matter). And third, it takes class time away from other, more traditional activities, such as my lecturing on textbook material.

Paired with each of these concerns, though, is a benefit. First, *I* am more engaged in *thinking* about the calculus than I have ever been in the past. I have been especially struck by the importance of viewing a function as continuous, in which case the rules for derivatives and integrals apply, or discrete, so that average rather than instantaneous rates of change make sense. Second, it seems that my students have done more thinking about calculus on their own, based on their written projects, than have students in past classes, based on less thoughtful responses to original application questions. And third, scores on mechanics-based exams involving derivatives and integrals have been at least as high as those from past years, even though I have spent less class time lecturing on and going over these processes. Perhaps the greatest benefit, though, is that (hopefully) most of the students in this course will have a pretty good answer the next time someone asks them, "When am I ever gonna use this stuff?"

Problems that matter: Teaching mathematics as critical engagement.

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INTRODUCTION

This paper is the result of a struggle to understand what it is about what I do that *really* matters. It started two and one-half years ago when I first became a high school mathematics teacher in Miami, Florida. More specifically, it started with the first of countless times my students asked "Mr. Bohl, why do we have to learn this?" This question has since become the focus of my thinking about mathematics education. In two years of teaching (including summers) I had the opportunity to work with a broad range of students, from classes of advanced college-bound students to a program for ninth-grade repeaters. The question 'Why?' came largely from students in the lower tracks, and I found it disconcerting that I could often not answer this ultimately important question. Ironically (or perhaps not?) those who were succeeding in school, the 'best students,' didn't seem to be as concerned with 'Why?' I wanted all of my students to be inclined to ask 'Why?', and I believed that they all deserved a satisfactory answer whether they asked it or not. This paper is an exploration of how we might reconfigure mathematics education to answer the 'Why?' question for all math students.

STANDARD ANSWERS TO THE 'WHY?" QUESTION

As time passed, I became increasingly frustrated with the onslaught of unanswerable 'Why?'s, the irrelevance of the materials available to me, and my very spotty success designing more relevant curricula. So I began asking other math teachers why they thought certain subjects and concepts were taught. Their answers fell into four categories: tomorrow, jobs, general mental strength, and tests. In the first category are answers of the type: "because they'll need it for tomorrow," "...for the next chapter," "...for next year," etc. When pushed further, this often resulted in "they'll need it for calculus." To some, then, we were teaching mathematics for the sake of other mathematics, with the ultimate goal being a year of calculus by high school's end. When I asked why students needed calculus at all, usually an answer in one of the other three categories was given.

The 'jobs' answer was obviously predicated on the belief that the best jobs require high levels of math. There can be no doubt that mathematics has economic utility. It often serves as a filter, or a base requirement, for jobs. Thus, those who do not succeed at a certain level of mathematics course work can be blocked from consideration for some jobs. However, the mathematics that most people actually use at work is probably taught by the middle of ninth grade. And the percentage of people who actually use calculus on the job is fantastically small. So, contrary to the myth, much of school mathematics has little actual vocational utility.[1]

The answer that 'mathematics improves general mental strength' is predicated on the belief that doing mathematics improves formal logical reasoning and problem-solving in a broad way. That is, mental processes are improved by math practice, and then become available for use in other situations — whether mathematical or otherwise. While mathematics training may or may not strengthen the mind generally,[2] it has been shown time and again that standard mathematics curricula are not good mathematics training. Most students lack facility with even the simplest real-life problem solving.[3] So here again what is being claimed has a questionable relationship to reality.

Now, I support math education that is vocationally useful, and that might make students stronger thinkers. However, the reality is that the path to calculus that we attempt to lead students down does neither of these well. There is one thing that this path does very well, however. It puts students into hierarchical lists that universities and employers use to simplify their choices of student and employee candidates.[4] When teachers respond to 'Why?' with the name of a test, it is because they understand the importance of a student's position in that queue. It is a very real and valid concern. In the bigger sense, however, if most students get little vocational or logical power from the ways we currently teach mathematics, then the way that mathematics matters most now is as a major part of our society's publicly-funded human sorting service. And even though some do benefit from the system, all students are cheated of the potential critical power that a strong mathematics education might help them develop.

I do not mean this exploration of answers to 'Why?' as an attack on teachers. The assumptions we educators call upon to justify our practices partly mirror those of school systems themselves, of which we are all products. I started realizing the power of these assumptions while muddling through attempts to develop curricula that I felt might matter to my students in ways other than as a sorting device. This led me to consider my own assumptions about why mathematics is important for students to study, and to imagine how my practice might be brought more in line with those assumptions.

WHY DOES MATHEMATICS MATTER?

My beliefs about why math is important are informed by three basic ideas. The first is that mathematics can give students a powerful way to relate to the world, not the mythical world of future jobs where they will utilize calculus, but rather their immediate world — the world that they actually inhabit during the time they are students, and that they will continue to inhabit after graduation. Students deal with situations, concerns, and activities every day that are rich with mathematics. They are bombarded with numbers from jobs, stores, ad agencies, the government, etc. Mathematical knowledge can be used to help students analyze and raise questions about such numbers and their implications, as well as to use numbers to understand the world in different ways.

The second reason — simply an extension of the first — is that mathematics knowledge is necessary for full participation in our democracy. The one way that mathematics knowledge (or lack thereof) will bear directly on the life of every student is in her/his role as citizen. Numerical data and mathematical models are integral parts of our reality.[5] They are used every day to decide such things as how many Americans need to be kept unemployed to ensure a "healthy" economy and how many dollars a human life is worth to an insurance company. Those who make such decisions wield great power to shape the reality that we all experience. In highly technical societies such as ours, mathematical competence is a major portion of democratic competence.[6] Math is increasingly used as a means of developing technology and directing public policy.[7] Even though most people will not use advanced math on their own jobs, all people need to be prepared to evaluate the work of those who make such decisions and to engage with the mathematical aspects of important social issues.

The third reason — a further extension on the theme — involves the relationship between mathematics and rational thought. Thanks in large part to Descartes, rational thought is widely accepted as the only worthy mode of cognition in western societies.[8] As a result, rational argument is, at least theoretically, the only accepted mode communication in public debates. Being able to rationally justify positions is a skill needed for individuals to gain public validity for their ideas. Thus, it is critical for citizenship. There is a clear tie between rational argument and the logical justification of mathematical results. Indeed, deductive mathematical proof is considered the purest type of rational argument. While the two are not equivalent, there are similarities that could be capitalized on by honing students' understandings of the specific structures of deductive logic. Thus, mathematics education might further enhance students' power as citizens by helping them make and critique rational arguments.

These overlapping justifications for my job, which will be expanded on in the following sections, have brought me to believe that we should teach mathematics that matters in two senses: it should matter to students and their immediate lives, and it should matter to the imperative of democratic citizenship.

MATHEMATICS THAT MATTERS TO STUDENTS

I believe that we need to help students learn to engage mathematically with their immediate worlds. Traditional mathematics curricula have not been successful at doing this because math is traditionally taught in a largely formal way. That is, it is taught without reference to the objects of people's real experiences. Such teaching stems in part from the beliefs that math is, by it very nature, abstract, and that it is math's abstract nature that allows it to transfer — or to be used — across a variety of concrete, real-life situations. This conception of transfer has been called into question,[9] and there has been some movement away from strictly formal learning in current reform trends.[10] However, most mathematics is still taught in ways that are artificial to students. Even curricula that claim to be 'realistic' are not sufficient. There is a gulf between teaching 'realistic' mathematics -which is word-problem- and situation-based — and 'real' mathematics, which actually involves the lives and interests of the students in the classroom.[11] It is through 'real' mathematics that I believe teaching should take place.

Mathematics education based on a real problem curriculum would directly involve students in exploring their worlds with math. We know that students have interests, but they normally do not become part of mathematics classes. Bringing students' lives into class can help on a very basic level: learning can happen far more easily when students see the direct relevance of what is being learned.[12] My experience has been that, when students' contexts were being studied, students involved themselves more actively, and I could concentrate on their intellectual development rather than on behavioral manipulation.

Since I believe that school mathematics should be geared toward helping people interact with each other and the world, this* is *a call to teach entirely applicable mathematics.

Teaching mathematics based on real contexts and situations familiar to students serves another important purpose. It allows curriculum to respect and capitalize on the rich collections of personal and cultural knowledge that students come to class already possessing. Schooling generally disregards and devalues much of students' personal and cultural experiential knowledge.[13] And, because of differences in their relationships to the dominant school culture, students from non-dominant cultures are especially mis-served by schools.[14] Opening the starting points of mathematical explorations to the concerns and interests of students can allow math teachers to become part of the remedy to this situation by broadening the bases of curriculum to include students' lives.[15] Multiple interests, concerns, and viewpoints could then be allowed a place in classroom discussion. Of course, all students do not share interests, concerns, and viewpoints, and opening the classroom to a multiplicity of voices invites in 'negative' along with 'positive' influences. This can greatly complicate classroom interaction.[16] However, since such complications are part of the reality with which I would like to help students engage, I prefer to incorporate them into, rather than exclude them from, the classroom discourse.

Teaching such mathematics would involve starting with particular real situations of interest to the students, and mathematizing them. Mathematizing involves gaining understandings about real situations by using mathematics.[17] Pedagogically, I like to think of it both as using math to uncover patterned relationships, and as imposing mathematical order on unordered realities. So to mathematize means analyzing a real situation either by mathematically modeling its components, or by quantifying its characteristics with statistics. By mathematizing the life contexts of particular students, and by using such mathematizations as the bases for learning, it becomes possible to inform the mathematics with the ideas and cultural constructs that students already possess.[18] This is not a call for a curricular add-on, but rather for a deep shift in our thinking about the relationship between mathematics and people's lives. Such a shift might happen if we take the lives and world views of all students seriously.[19]

I can hear the formalist questions arising: "What kinds of mathematics can be taught this way? That is not mathematics at all, but mathematics applications."[20] My answer to that charge is: well...yes and no. Since I believe that school mathematics should be geared toward helping people interact with each other and the world, this *is* a call to teach entirely applicable mathematics. However, that does not mean that mathematics need never be addressed at the formal level. There is a great variety of mathematics that can be soaked from and used to analyze even the simplest real situations. And there would no doubt be times, as multiple real contexts are mathematized, that formal mathematical issues would need to be addressed.[21] It is, after all, the exploitation of similarities of pattern across situations that gives mathematics its power. So, in attempts to help students comprehend that power, there would need to be explorations of the similarities between the patterned aspects of different real situations. That is exactly how much of mathematics was historically developed in the first place.[22] So such abstraction and pattern seeking should certainly continue as one goal of mathematics education. My point is that mathematical abstraction should not be viewed as the only goal of mathematics education. Its importance needs to be reconsidered.

What I am arguing for is that mathematics be taught through mathematizing (or 'making mathematical') rather than through concretizing (or 'making concrete').[23] Traditional school mathematics, especially from the onset of algebra, introduces concepts and objects at the abstract level, and then gives concrete examples of them. That is, it starts with abstractions and concretizes them. What we need to do is reverse that priority, put concrete examples in the foreground, and build abstractions from those through mathematization.[24] Now, as contexts are mathematized, and as teachers and students work to formalize some of the mathematics that arise from those mathematizations, certainly some concepts that are currently taught would fail to show up. In those cases, I suggest we would need to rethink their curricular importance. That is not to suggest that concepts that don't arise from students' situations and interests should never be taught in schools. However, with a curriculum that is driven by calculus — a math that few people would ever have reason to use in real life — it is obvious that we need to rethink our curricular priorities. The dilemma posed by mathematics that are absent from mathematizations of reality would make a fine starting point.

By starting with the concrete and then moving to the abstract, we side-stepped a tension that arose earlier when students that I first taught abstractly tried to 'apply' the mathematics.

I learned the value of using students' contexts as the basis for teaching through two experiences teaching the graphing of points, lines, and functional relationships in the x-y plane to pre-algebra students. The first time through was with a group of middle track students. I started with traditional methods, including a few graphing games, to familiarize the students with plotting in the abstract x-y plane. Then we used tables and simple calculations to graph patterns and lines. We finished up with activities that were a bit more realistic, using function-like calculations to answer questions based on local bus travel and other familiar situations. What surprised me is that the concept of plotting points never seemed to make sense to most of the students, and the simplest graphing was a haphazard undertaking even at the end of the nearly two-week-long unit. And since they didn't understand how to graph in the x-y plane, graphing on different types of axes (for instance graphing *time worked* and *dollars earned*) was very difficult for most to grasp.

The next time I taught this was with students in a special program called School Within a School (SWAS) for the school's repeating ninth-graders. We happened to be in the middle of hurricane season and hurricane Opal had just done a dance around our end of the state. To introduce graphing, we started with hurricane maps. The coordinates to plot the location of the hurricane du jour were published daily, along with information about wind speed, velocity, and direction of travel. We first learned how to find locations by plotting Opal, by plotting and reading the plots of several fictitious hurricanes, and by exploring the daily changes in direction and speed and how those appeared on the graphs.

From there, we tied the ideas of latitude, longitude, and compass direction into the structure of Miami's grid-like street map. In Miami, all east-west roads are called streets and north-south roads are called avenues, and each is sequentially numbered starting with zero downtown. This means that the street address of any building includes all the information needed to go there. Now, all of my students knew exactly how to find places by their addresses. We used their knowledge of the address system, along with the similar situation of hurricane mapping, to develop an intuitive understanding of the abstract idea of plotting points in the x-y plane. With that solid base, we continued on to explore tables of values and graphs of functional situations. This time the students, who for the most part had long records of poor math performance, didn't flinch when we switched from plotting (x , y) points to forms such as (hours worked, dollars earned). By that time they were familiar with several types of points, e.g. (latitude, longitude) and (street name, street number). By starting with the concrete and then moving to the abstract, we side-stepped a tension that arose earlier when students that I first taught abstractly tried to 'apply' the mathematics.

Another important benefit of the fact that we were exploring their real world was that several discussions arose about differences between different areas on Miami's map. Although I didn't capitalize as much as I might have on discussions of disparity of wealth in different areas, we did touch a bit on that issue as well as others. Had I been more experienced, I'm sure I would have been able to weave such issues more tightly into the content. On the other hand, had I not used the students' real context as a base for teaching the topic, such issues would never have surfaced at all.

MATHEMATICS THAT MATTERS TO THE DEMOCRATIC IMPERATIVE

Teaching mathematics that matters to students' immediate lives is important. However, it also needs to be taught in ways that matter to the democratic imperative. I believe that all education should help improve students' qualifications as citizens, and mathematics is no exception. Many mathematics reformers, including the NCTM, acknowledge that mathematics knowledge is important for an informed citizenry.[25] What a functioning democracy needs, however, is not simply informed citizens. Because democracy is, in theory, about self-government, it requires active involvement to function correctly. What we need, then, are both informed and engaged citizens, who can engage intelligently with societal issues and debates once they become informed of them. Traditionally, the teaching of school subjects does not encourage such engagement.[26] Mathematics, with its ethos of abstract detachment, tends to be the worst offender of all on this measure. And, as mentioned above, there are numerous ways in which mathematical competence is necessary for intelligent engagement with today's important social issues.

If one accepts that we should teach mathematics by mathematizing students' contexts, the next question is: how can we make such mathematics democratically important? Doing that requires, whenever possible, mathematizing situations that involve socially relevant issues that students can engage with.[27] It also requires using the process of mathematical engagement as the basis for making judgments and taking actions based on those judgments. This can help prepare students to become the confident question posers, problem solvers, and mathematical/rational communicators that our democracy requires.[28] In this sense, mathematics classrooms can serve as places where students actually enact democratic principles through the practice of democratic citizenship. An example of this is another unit I taught to my SWAS ninth-graders. This unit was designed to introduce both descriptive and inferential statistics, and was a first attempt at fully enacting my beliefs about the need to teach both personally and socially relevant mathematics.[29]

Inferential statistics is the statistics through which inferences are made about entire populations from a few representatives of the population; unfortunately, it is not generally taught in K-12 curricula. This unit was designed to teach it because inference is what gives statistics nearly all of both its strengths and weaknesses. It is a critically valuable mathematics for citizens, but is usually never seen by students, especially like those in SWAS who were not bound for any mathematics beyond geometry or Algebra II.[30]

We started with reading statistical graphics from newspapers and discussing the information they represented, the questions that could be asked about them, and the means by which the information may have been gathered. We then created a survey to be taken anonymously by members of the school's student population. The choice of survey questions was left to the students. This helped gain great interest, and resulted in some of the most engaging discussions we'd had all year. The data we gathered, along with the data from several smaller surveys the students administered, were analyzed as we explored the ideas of populations, random selection, and inference; created graphic representations of the data; and discussed how confident we could be about our inferences.

As a culminating unit project, each student had to create and administer her/his own one-question survey on a topic s/he felt was socially important. This involved deciding on a target population, determining how to obtain a random sample of it, and then writing up a short report with mathematical justifications for the inference made. The students were also to share that report with someone in a position of authority who they thought should be familiar with the knowledge they had created.

We experienced all the problems attendant with teaching something the first time, the end-of-the-year jitters, and working with students that the school system had miserably failed. Even so, the unit was as mathematically successful as anything else we had done. This itself was a victory given that the mathematics we were dealing with was of a much higher level than usual.

Regarding teaching mathematics that matters to students, we worked with topics that were of immediate concern to them. We used the mathematics we were learning as a means of gaining deeper understandings of their immediate surroundings. This was not a "teach them now so they'll know how to use it later" approach. Rather, it was "learn as you do." The real-life basis for what we learned allowed us to have very thoughtful discussions about student interests and concerns and gave us a place to ground the more abstract mathematics we were exploring.

In terms of informed and engaged citizenship, simply mulling over the data we collected made the students aware of things they'd not known before about their environment. We used the data to engage in discussions about how statistics might be used to deceive and what the requirements for making valid inferences are. Much to my dismay, bad planning meant that students did not have time to report their findings to a figure of authority. However, the requirement to do so did actively engage many of them, including several who otherwise had shown little interest in the class all year. Many designed their surveys to address questions of specific relevance to certain authorities. As examples: one surveyed the student body so she could let the new principal know how students felt about his first year's performance; another surveyed pregnant teens in her housing complex about reasons for getting pregnant so she could report it to the school's health clinic counselors to help them better counsel girls about pregnancy; and a third surveyed male students to find out how they felt about teen fathers' role as parent so he could inform the guidance counselors of males' thoughts on the topic.

This unit offered the students a small experience with creating knowledge about something that concerned them, and putting it in an 'officially sanctioned' form that allowed them to participate in the discourse of authority. All students deserve to have such opportunities, and the imperatives of our democratic technological society demand that they do. Given that mathematics is a major part of our society's official mode of discourse, students must have experiences where they learn to be comfortable utilizing mathematics to communicate within it.

RATIONAL ARGUMENT AND MATHEMATICS

In our highly rational and scientific world, mathematical logic serves as the prototypical means of finding and proving truth. It is also very closely related to the rational language of the public sphere. Helping students master the art of logical argument has long played a role in mathematics education. Geometric proofs have usually served as a means of introducing students to this art. However, this is one area of mathematics that is absolutely never related to the real world. Proofs are the ultimate source of mathematics' assumed power of abstraction and generalization, but they never refer to things that actually exist, only to abstract mathematical objects.

Being able to rationally justify one's positions is an important part of gaining validity for one's thoughts in our rationalistic society. It was with this in mind that I made the decision not to teach formal proofs when I taught geometry. Instead, we focused on learning how to write coherent and rigorous paragraph justifications for solutions to specific problems. My feeling was that if students could learn the structures behind written justifications for specific solutions, they would be a step ahead in the work of learning to rationally justify other thoughts as well. In my classes many students excelled at this, even some of whom were struggling in other areas. Such justification should not wait until geometry, however. It should start in the earliest years of grade school mathematics.[31] In the statistics unit with my SWAS students, we did focus on what knowledge was necessary to be able to confidently make an inference. This was a small but necessary step in learning the need to justify statements.

My feeling was that if students could learn the structures behind written justifications for specific solutions, they would be a step ahead in the work of learning to rationally justify other thoughts as well.

One mistake I made while teaching paragraph justifications was that we never explored the real-world implications of rational thought in reference to objects other than mathematical objects. A key to mastering both mathematical deduction and rational justification is understanding that what you can conclude depends entirely on what you assume. Mathematical work is based on logical movement from a set of assumptions to a set of conclusions. Because of the strict logical structure of mathematical argument, the assumptions entirely dictate what can be concluded.

In everyday rational argument, similar logical rules apply, at least theoretically. However, in everyday argument assumptions are not usually the focus of scrutiny. Often, when someone does or says something that doesn't make sense to us, we take for granted that the speaker/actor is not making sense at all. This presumption is very often wrong. Often, when someone's reasoning doesn't seem rational or reasonable, it is simply because s/he is working from a different set of assumptions. As with mathematics, two people can make perfectly sensible arguments that result in opposite conclusions if the assumptions they make are different.

Much misunderstanding between different individuals and groups of people in our society results from inattention to assumptions. Learning to question the assumptions that allow people, including ourselves, to arrive at the conclusions we do is an important step in being able to take a critical role in rational discourse. Mathematics is, in part, the art understanding this relationship between assumptions and conclusions. Thus, the mathematics classroom seems like a fine place to involve students in learning to question the assumptions they and others make about issues and problems that matter.

QUALIFICATIONS ON MATHEMATIZATION

So far I've argued that we need to teach mathematics through the mathematization of real, socially relevant situations. There are certainly benefits for citizens of our highly technical democracy to learning through and about the processes of mathematizing. However, mathematization should be not be uncritically adopted as an all-encompassing mode of analysis or understanding. Students need to be exposed to these modes of analysis not only so that they can use and learn from them, but also so that they can take an active part in critiquing them.

Mathematics is only one of many ways of making sense of the world, and it is probably our most morally vacant. Much has been written on the negative effects of the detached, positivist, and essentialist modes of interpreting the world that have developed along with our ability to quantify and categorize.[32] The authority of numbers is pervasive in our society, and they are often used to gain authority for misleading, and even untruthful, analyses.[33] Earlier I described mathematizing as "imposing mathematical order on unordered realities." There are situations where such order helps us understand things we might not have otherwise. However, mathematizations can just as easily be used to bad ends as good. The determining factor is the set of assumptions made in the process of mathematizing.

As already discussed, assumptions largely determine conclusions, and with the complexities involved with mathematizing, there is certainly room for disagreement about which assumptions should be made and which should not. A prime example is the already mentioned SAT. Because I assume that achievement is not illustrated by one's ability to answer multiple-choice questions, I would not attempt to mathematize it by means of such a test (and perhaps not at all). Those who publish the test obviously make a different set of assumptions to arrive at the conclusion that the test is valid — a conclusion which they use statistics to 'prove.'

Students need to be made aware of the ways that math can harm as well as how it can help. In accepting mathematizing as a mode of mathematics instruction, it would be critically important to also accept the critiquing of mathematizing as part of that mode. Where would such issues be addressed if not in mathematics classrooms?[34]

CONCLUSION

Many people, including my former students, feel that school mathematics is irrelevant to their lives. This is not the result of their inability to comprehend reality. It seems to be simply a common sense recognition of the fact that, as it is currently taught, mathematics does not matter for most people except in its role as a sorting mechanism. We lead students to calculus when what they are exposed to in real life is statistics. This fact alone gives credence to people's questioning mathematics' relevance. If we want people to think that school mathematics is Important in their lives, we need to teach mathematics that actually is important.

These ideas are a preliminary sketch of ways we might reconfigure mathematics education so that it actually does matter in real people's real lives. The units described are some of my first attempts to do so. Obviously such approaches require a very different orientation toward mathematics than most of us math teachers, as products of schools ourselves, are familiar with. Hans Freudenthal, who pioneered work in the area of 'realistic' mathematics education, wrote that "Mathematics is an activity, a behavior, a state of mind... an attitude, [and] a way of attacking problems."[35] In order to take Freudenthal seriously, we need a more open-ended approach to mathematics education that requires deep involvement with real problems rather than simply the acquisition of skills that are never applied to real problems. As mentioned, however, realistic mathematics is not sufficient. Even if we approach mathematics as "an attitude, [and] a way of attacking problems," the question remains, whose problems are worthy of consideration?[36] Mathematics that is based on either abstractions or pre-defined general types of situations fosters classroom atmospheres that lock out students' experiences, concerns, and cultural backgrounds. It also locks out discussion of the types of issues that matter to our democratic society. We need to help students understand how to critically engage with the world mathematically, as well as how to engage critically with mathematics.

Mathematics educators need to have a good and honest answer to the question: Why do we need to learn this? If we want school mathematics to matter, we need to teach it as active engagement with problems that matter — both to students and to our democracy. If we don't care that school mathematics matter, we could continue teaching it based on the imperatives of calculus. Then we could rest assured that, in the future as now, much of the mathematics taught will not matter, and much of the critical mathematical power that could be developed in our society will lay dormant.

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REFERENCES

[1] John P. Smith. III. "Mathematics in Michigan's workplaces." Presentation made at Michigan State University's Mathematics Education Seminar, (1996, May).

[2] Jean Lave, *Cognition in Practice*, (Cambridge: Cambridge University Press, 1988).

[3] This was illustrated in 1983 with the Third NAEP Mathematics Assessment test by the National Assessment of Educational Progress. For sources of further discussion, see Edward A. Silver & Judith S. Zawojewski, "BOSUN Project Progress Report." Unpublished Research Report, University of Pittsburgh, (1996).

[4] Michael W. Apple, "Do the Standards Go Far Enough? Power, Policy, and Practice in Mathematics Education." *Journal for Research in Mathematics Education* 23 (Number 5, 1992), 412-431.

[5] Marilyn Frankenstein, *Relearning Mathematics: A Different Third R — Radical Maths*. (London: Free Association Books, 1989) and Ralph T. Putnam, Magdalene Lampert, & Penelope L. Peterson, "Alternative Perspectives on Knowing Mathematics in Elementary Schools." In C. B. Cazden (Ed.), *Review of Educational Research*. (Washington, D.C.: American Educational Research Association, 1990), 57-152.

[6] Ole Skovsmose, "Mathematical Education and Democracy." *Educational Studies in Mathematics* 21 (1990), 109-128.

[7] See Apple, "Do the Standards Go Far Enough", where Apple describes mathematical knowledge used in this manner as "technical/administrative knowledge," the control of which he posits is concentrated in the hands of a limited group of people. Making the distribution of that control more equitable is, in large part, the goal of the type of teaching described here.

[8] Putnam, Lampert, & Peterson, "Alternative Perspectives on Knowing Mathematics in Elementary Schools."

[9] Jean Lave, *Cognition in Practice*.

[10] For an example, see National Council of Teachers of Mathematics, *Curriculum and Evaluation Standards for School Mathematics*. (Reston VA: Author, 1989).

[11] William F. Tate IV, "Mathematizing and the Democracy: The Need for an Education that is Multicultural and Social Reconstructionist." In Carl A. Grant & Mary Louise Gomez (Eds.), *Making School Multicultural: Campus and Classroom*. (Englewood Cliffs, NJ: Prentice.-Hall, 1996), 185-201.

[12] Rodney E. McNair, "Context and Autonomy." A paper presented at the annual conference of the National Council of Teachers of Mathematics, Minneapolis, MN (1997).

[13] Stanley Aronowitz & Henry A. Giroux, *Education Still Under Siege*. (London: Bergin & Garvey, 1993).

[14] Lisa Delpit, *Other People's Children: Cultural Conflict in the Classroom*. (New York: The New Press, 1995).

[15] See Gloria Ladson-Billings, *Dream Keepers*. (San Francisco: Jossey-Bass, 1994). Here Ladson-Billings discusses the importance of including students' lives in the classroom.

[16] Deborah Loewenberg Ball, "Transforming Pedagogy: Classrooms as Mathematical Communities. A Response to Timothy

Lensmire and John Pryor." *Harvard Educational Review* 65, (1995), 670-677.
[17] Ole Skovsmose, "Mathematical Education vs. Critical Education." *Educational Studies in Mathematics* 16 (1985), 337-354.
[18] McNair, "Context and Autonomy."
[19] See Tate, "Mathematizing and the Democracy," 187-190. Tate gives an example that makes the need for this shift clear.
[20] Robert Thomas, "Proto-mathematics and/or real mathematics." *For the Learning of Mathematics* 16 (Number 2, 1996), 11-18.
[21] See Robert P. Moses, Mieko Kamii, Susan McAllister Swap & Jeffrey Howard, "The Algebra Project: Organizing in the Spirit of Ella." *Harvard Educational Review* 59 (Number 4, 1989), 423-443. The authors offer a fine example of using the experience of riding the subway to help students develop deep understandings of the abstract concept of negative numbers.
[22] See Anna Sfard, "On the Dual Nature of Mathematical Conceptions: Reflections on Processes and Objects as Different Sides of the Same Coin." *Educational Studies in Mathematics* 22 (1991), 1-36, Here Sfard gives short descriptions of the development of both the number and function concepts.
[23] Skovsmose, "Mathematical Education and Democracy," and Bill Rosenthal, "No More Sadistics, No More Sadists, No More Victims." *UMAP Journal* 13 (Number 4, 1990), 281-290.
[24] Christine Keitel, "What Are the Goals of Mathematics for All?" *Journal of Curriculum Studies* 19 (Number 5, 1987), 393-407.
[25] National Council of Teachers of Mathematics, *Curriculum and Evaluation Standardsfor School Mathematics*. (Reston VA: Author, 1989).
[26] Henry A. Giroux & Roger Simon, "Popular Culture and Critical Pedagogy: Everyday Life as a Basis for Curricular Knowledge." In Henry A. Giroux & Peter L McClaren (Eds.) *Critical Pedagogy, the State, and Cultural Struggle*. (Albany NY: State University of New York Press, 1989), 236-252.
[27] In "Mathematical Education and Democracy," Skovsmose discusses the seeming contradiction between teaching based on students' interests and teaching based on the democratic imperative.
[28] Tate, "Mathematizing and the Democracy."
[29] The statistics unit described was designed in collaboration with Bill Rosenthal. The collaboration was part of our ongoing efforts to understand how to best bridge the divide between critically-oriented university academics and progressive classroom teachers.
[30] Bill Rosenthal, "No More Sadistics, No More Sadists, No More Victims." *UMAP Journal* 13 (Number 4, 1990), 281-290.
[31] National Council of Teachers of Mathematics, *Curriculum and Evaluation Standards for School Mathematics*.
[32] See, for example, P. J. Davis & R. Hersh, *Descartes' Dream: The World According to Mathematics*. (New York: Harcourt, Brace Jovanovich, 1987), cited in Putnam, Lampert, & Peterson, "Alternative Perspectives on Knowing Mathematics in Elementary Schools."
[33] Stephen J. Gould, *The Mismeasure of Man*. (New York: W. W. Norton, 198 1), cited in Tate, "Mathematizing and the Democracy."
[34] Terezinha Nunes, Analucia Dias Schliemann, & David William Carraher, *Street Mathematics and School Mathematics*. (Cambridge: Cambridge University Press, 1993).
[35] Hans Freudenthal, IOWO- Mathematik Fur alle und Jedermann. *Neue Sammlung* 20 (Number 6, 1980) 634-635, cited in Keitel, "What Are the Goals of Mathematics for All?"
[36] Michael W. Apple, "Do the Standards Go Far Enough? Power, Policy, and Practice in Mathematics Education." *Journal for Research in Mathematics Education* 23 (Number 5, 1992), 412-431.

Platonism and All That...

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own research: yes, proofs I have invented, but the patterns which the proofs legitimate seem to have been there, waiting to be found. I have no idea what absolute reality is like, but I can tell you what it felt like to find these things.

And, so, back to Plato and his cave; the firelight casting shadows on the wall. We face the wall, and guess, if we will, what makes the shadows. Sometimes mathematics seems firm, unshadowlike. But sometimes the shadows waver. In *Proofs and Refutations*, Lakatos (1976) documents the wavering which may take place. He says we never know whether our proofs are right, but he believes we can be sure of their improvement. And what of Gödel? Undecidability promises that we will never come to the end of our search, because the choice amongst the undecidables will remain, and the absence of a consistency proof is the guarantee that shadows, not ultimates, are what we see. I think I am a Platonist at night.

Book Review: *The Courage To Teach* by Parker Palmer

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The Courage To Teach. Parker Palmer. Jossey-Bass Publishers: CA, 1998. ISBN 0-789-1058-9

Parker Palmer is a self-described writer, teacher, and activist "who works independently on issues in education, community, leadership, spirituality, and social change." The words "social change" are not likely to be heard reverberating in the halls of mathematics departments, much less, "spirituality," and since nothing in this book pertains directly to the teaching of mathematics (insofar as it only appears in one brief paragraph in this book), it is likely that this book will never find its way into the hands of many individuals in mathematical circles who might benefit from it and enjoy it. In traveling the country giving workshops on teaching, Palmer, to be sure, is well acquainted with the particular problems posed by the teaching of mathematics and science. On the other hand, he is not about to discuss how to teach any specific topic; not only would this go contrary to his grain to tell us how things should run, but it would run counter to his premise that good education in all fields (or good learning) shares common attributes, and that learning occurs best when all are engaged in that process—faculty as well as students—within a learning "community" considerably broader than a mathematics department.

The effect of his engaging with teaching in broad terms is that I can interpret what he says in ways that apply meaningfully to myself as a teacher of my students in my classroom in my department, with my thoughts about mathematics as it is now perceived and used in the world. And frankly, so exhausted am I in this mathematician's environment, in which the discussions over the internet and within departments range in emotion from mild to extreme forms of anger (over calculus reform, redesigning courses, discussing attrition with the engineers, and playing the "shell game" in Palmer's words, of whether student evaluations will or will not count) that it's doubtful I could have suffered through a book that found teachers deficient. This book has the opposite effect: it is affirming and understanding, and I say this as one whose cynic-button tends to light up when the word "heart" is used in conjunction with teaching. But because the language comes from an honest place, and because his insights, based on years of discussion with teachers across the country, are quite gentle and beautiful, this book has much to stimulate one's thinking about teaching.

One might ask, and I did: am I succumbing to feel-good psychologizing or isn't this spiritual thing a bit thick—is there too much of the "thee/thou" in this book? (Palmer is a sociologist with a year in a theological seminary who has taught at a Quaker school.) But turn the question around to another of equal validity: doesn't this language make as much sense as any we have available to us to discuss an entire realm of sociological problems that occupy us as mathematics teachers: i.e., sociological issues regarding unprepared students, non-supportive administrations, and so forth? Educational philosophers such as Dewey, C.S. Peirce and other American pragmatists were willing to accept the sociological makeup of where-we-are-now as breeding grounds for generating philosophical ideas about learning; why should we not be prepared to listen for signals from that breeding ground? Annie Dillard wrote the book "Teaching a Stone to Talk," which of course, Palmer says, is about learning how to listen to rocks. Why shouldn't we be able, at least, to listen to how others are discussing education around us?

The chapters in this book build from the inner to the outer aspects of our jobs, from examining the "heart" of the teacher (identity and integrity) and the "culture of fear" which academia seems to foster, to examining the larger community in which we teach. We are led along in our thinking by questions and paradoxes, rather than prescriptions and answers. As the poet Rilke says, we must learn to "love the questions themselves;" holding the tension thus, "makes the heart larger." The author's style of making his points using dialectic, dichotomies, and paradox is really very captivating. For example, he writes that the teaching space should be bounded and open, hospitable and "charged," able to hear from the individual and the group, allow "little" stories as well as the "big," support solitude and surround it with the resources of the community, welcome both silence and speech.

We learn best when we "hold the tension of opposites." (I am reminded of the cliché: "one idea speaks, second idea argues, third idea presents itself and is good.") Eventually we evolve into thinking about larger issues, to consider not just the classroom in which we teach, but beyond that, to how education in general terms is at its best when it is a process.

As mathematicians, we might well stop right here: "What or who is our mathematics community?" Many in our discipline are, using Palmer's word, "purists"—Platonists who now find ourselves being hammered into serving the "client disciplines" or functioning as watchdogs for remedial/developmental mathematics. The single reference to mathematicians in this book is in the form of a quote in *The Chronicle of Higher Education* from an unnamed mathematician. He says:

> Our preliminary responsibility as mathematicians is not to students but to mathematics: to preserve, create, and enhance good mathematics and to protect the subject for future generations. Good students [the ones destined to become mathematicians] will survive any educational system, and those are the ones with whom our future lies.

This is a terrible point of view for a teacher to hold: that the subject must be protected, presumably even from students. There are more warnings in this book to be wary of objectivist styles of teaching. Alfred North Whitehead declared that objective, "inert" ideas are the bane of higher education, deadening the process of teaching and learning for students and teacher alike. On the other hand, if this mathematician's quote means what I think, this professor is expressing a disappointment and frustration that many of us would share—that centuries of collaborative human endeavor in building up a foundation of rigor that would move us as close as possible to workable truths, is currently slipping away into muddied technological approximations and being blurred by methods of teaching that shortchange the subject so as to pass students through. That the ways in which mathematics is taught are grinding down this field. Palmer uses the example of this quote from the Chronicle to illustrate abuse of teacher-centered models of teaching, and elsewhere hints at an academic pecking order in which science would be on top: "...every 'soft' discipline in the curriculum has practitioners doing research that is more objectivist than thou: literary scholars who count adverbs rather than explore meanings, psychologists who analyze the data of human behavior as if people had no more inner life than Styrofoam." Nevertheless I am confident that in a workshop setting, Palmer would probably enjoy the difficulties of trying to identify the paradoxes of this teacher's experience, ultimately to lay that problem and others at our feet as responsibilities for our educational community to struggle with, the struggle being part of our learning experience.

If a community of learning is what Palmer requires, he identifies four ways in which the community can be constructed: (1) community as a business-oriented enterprise (as with Total Quality Management, in which students are the "customers"— although they're not always right), (2) community as a therapeutic organization (which must address the wounds of the injured), (3) community as a civic structure (with governing, hierarchial roles and rules and conventions), and finally—the only role he endorses, (4) a community of learners and knowers in a subject-centered educational environment. Thus the mathematician whom he quotes above has his place, too, in this type of community.

Following the book backwards, what does the community have to say about the smaller world of the classroom? To create a functioning educational community, the top-down model of teacher imparting wisdom to students will not work well because there are "baffles" to the learning on the way down, and these baffles not only constrain what can trickle down, but can jam and cause the flow to back up. ("We don't care if civilization goes down the drain as long as it doesn't back up.") Rather, we must have subject-centered classrooms; teachers and students share their views on the subject—and one should imagine here a diagram of a complete graph with "subject" in the center. In Robert Frost's words, "We dance round in a ring and suppose,/But the Secret sits in the middle and knows."

Palmer does not advertise any methods for the classroom; he briefly mentions group work and expresses curiosity about interactive methods—he enjoys astronomy software that allows him to feel a part of the universe—but he resists giving answers and solutions. In fact, if he would promote any technique for teaching at all it would be to ask questions and...wait, ask more questions and...wait. He adds many personal anecdotes that speak to his own frustrations in teaching; in this way, I am reminded of the words in Frost's epitaph, "He had a lover's quarrel with the world," although of Palmer, we might say, "He is looking to make up that quarrel." In his workshops, to get discussion going, he often asks teachers for a "critical moment" in their teaching. The responses range from positive to negative, but the shared discussions offer participants a sense of the mutuality of their experiences. (Another exercise is to fill in the blank in the sentence following, with the best possible metaphor: "When I am at my best teaching, I am like a _____". To my own shock, I might have answered nurse but Palmer himself was a sheepdog.) Other suggestions for a communal affirming of teachers are "clearness committees" for purposes of listening and sharing, and standards for the evaluation of teachers, linked to their ability to listen and change. Again and again in this book, one reads that dissension, disagreement, and frustration are the natural components of the learning experience, and not only should students know this, but we should be able to hear this from students with more composure and less fear. He says,

> If a space is to support learning, it must invite students to find their authentic voices, whether or not they speak in ways approved by others. Learning does not happen when students are unable to express their ideas, emotions, confusions, ignorance, and prejudices. In fact, only when people can speak their minds does education have a chance to happen.

All of this leads us down to the level of the individual teacher. (I deliberately unwound the book back to the point where Palmer begins.) As long as we teach with a sense of our own identity (characteristics—of who we are and what we feel) and integrity (character enough to be true to that identity), then we are well-equipped with the "courage" to enter a world that often "equates work with suffering" as guides, or authorities (authors) in that world, rather than as powers.

> The conclusion seems clear: we cannot know the great things of the universe until we know ourselves to be great things. Absolutism and relativism have ravaged not only the things of the world, but our sense of the knowing self as well. We are whiplashed between an arrogant overestimation of ourselves and a servile underestimation of ourselves, but the outcome is always the same: a distortion of the humble yet exhalted reality of the human self, a paradoxical pearl of great price.

So here is a book looking at education as a general process, without any particular nod (or bow) to those of us who in mathematics feel we are not understood for our need to operate with a separate set of rules. Do we now dismiss this book, or can we learn from it? Our mission in education is the same as that in other subjects. If the mathematics community is going to address the idea of being able to teach large numbers of students, is it going to be conceived of as being available to these people? This is not a direct challenge from the book, nor are there direct answers. But this book can provide some resources for ways of thinking about ourselves as others see us. I think this book would probably work very well as the basis for a mathematics teaching workshop. It could, at minimum, provoke us to reflect on ways in which we are part of the rest of the education community and ways in which we feel we are not. At base, the book would require us not to forget the subjective while we teach the objective; we have both an entitlement and responsibility as teachers to listen respectfully not only to the voices of our students but to the voice of our own "teacher within," a voice we would often prefer to muzzle. Palmer quotes Richard Gelwick, an interpreter of the chemist Michael Polanyi:

> Several times in public lectures, I heard [Polanyi] correct people who stood up to support him, [people who said] that they agreed that all knowledge had a personal element in it [and] then went on to say that this personal element was the risky part and that we should try to minimize it. Polanyi would explain that the personal was not to be minimized but understood as the element that was essential, the one that led us to break out and make new discoveries, and not at all an unfortunate imperfection in human epistemology. On the contrary, it is the cornerstone upon which culture, civilization, and progress were developed."

On Solving Equations, Negative Numbers, and Other Absurdities: Part I

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0. INTRODUCTION

Let me be clear about the development in school algebra I wish to track here. It is exemplified by the "solution of equations," as I learned it in my childhood around 1938. Before the era of The New Math, the setting up of linear and quadratic equations to model stories of perimeters and areas, and ages of fathers and sons,was generally done in the 9th grade. Having got an equation with x the unknown — this having been the hardest part — we would apply some rules such as that "Equals added to equals are equal", or maybe some rituals called "transposition" and "dividing through," to obtain one or more numbers we called the "solution," which we then "checked" by substitution. If the answer didn't check, one would look back for some miscalculation; otherwise we were done.

Most books — and many teachers, including my own — made little effort to put into English what we were doing. Algebra, it appeared, was a language and literature of its own, unconnected with words like "if," "then," or "but." Its pronouncements did not begin with capital letters or end in periods. It was no wonder that routine calculations like factoring were easy for us and "story problems" very hard. What can stories have to do with algebra?

It will be the purpose of most of what follows to work through a rather simple problem such as should be understandable to any beginner in high school algebra, in order to show how putting it into English makes all the difference between a ritual and an epiphany. Not that I advocate a *lesson* along these lines (it would take some weeks, I should think, and in part would have to stretch over years), but that I advocate a *curriculum* along these lines, or, if not a curriculum, a continuing conversation in algebra classes that conveys the lesson I hope to illustrate with this example.

The last few sections concern more sophisticated interpretations of this problem and a very similar one, which illustrate how mathematics of an unreal sort can get used in a real world, and how understanding of such usage would be impossible without full appreciation of the logic of the simpler versions.

1. A SIMPLE PROBLEM IN 9TH GRADE ALGEBRA

Here is a typical "story problem" such as might have been found in high school algebra in 1840 as easily as in 1997. Very likely this problem was known (and solved) in 1997 B.C. as well, in ancient Babylonia. No calculators are needed, and only the simplest arithmetic and algebraic notation enter.

<u>PROBLEM</u>: A rectangular garden is to have an area of 600 square yards, and its length is to be fifty yards greater than its width. What are its dimensions?

<u>SOLUTION</u>: Let x be the length; then x-50 is the width, and x(x-50) therefore the area. So:

$$x(x-50) = 600$$
$$x^2-50x-600 = 0$$
$$(x-60)(x+10) = 0$$
$$x = 60$$
$$x = -10$$

Now what? Well, -10 can't be the length of a garden, so the answer must be 60. We put a circle around the '60' and wrote, if we were meticulous, and the year was 1938, something like this:

"<u>CHECK</u>: Length x = 60
w = x-50 = 10
60•10 = 600, check."

In my day we got 10 points for this. What more is there to say?

A thoughtful student might wonder where that -10 came from and where it went so suddenly. "Length x = 60" we wrote; why not "Length x = -10?" If asked, the teacher might reply, "Well, that's not a length, is it?" Or, "The length can't be negative." Somewhere else in the book (fifty years ago; today it is no longer so) there might have turned up "extraneous roots," i.e. apparent answers to algebraic equations that for some reason didn't work; maybe this was such a case. Perhaps that was why we had to go through "checking the answer." We will come back to this later on. For the moment, let us consider the language in which the above solution was written. The standard format appears to be a string of equations without punctuation. Let us look again at the model "solution" as it typically appears (*verbatim*) in the student's notebook, and even as printed in many typical textbooks:

Solution: Let x be the length, so w = x-50, A = x(x-50).

$$x(x-50) = 600$$
$$x^2-50x-600 = 0$$
$$(x-60)(x+10) = 0$$
$$x = 60$$
$$x = -10$$

After the setting-up, "Let x be the length, so...", there are no commas, no periods, no words. The meaning of "x" was established at the beginning, and the rest seems to be equations, i.e. sentences so simple that periods aren't needed. But actually it is *not* clear that the meaning of x has been established, for all that it was said to be "the" length, since we seem to end with a sudden appearance of two answers, one of which (the negative number) gave little pleasure to either the textbook or the teacher. One might ask the teacher, of these last two lines in the student's notebook, does this mean "x = -10 OR x = 60" — or does it mean to say "x = -10 AND x = 60?" Neither interpretation seems to fit the idea that "x" had a *definition*, that is, a single meaning.

Yet that was the way it began: "x" was supposed to represent "length," a very definite length, a very definite number: the length of the garden wall, perhaps, or a prescription for the purchase of lumber. But isn't x also a "variable?" Maybe it is an "unknown." Is a variable or an unknown different from a number? This isn't funny. The amount of nonsense that has been written about "variables" has not only filled volumes, but has confused generations of both students and their teachers. One begins to suspect that the lack of punctuation and connectives such as "or" and "and," in the traditional way of writing the solution to this problem, are not just abbreviations, but evasions.

2. PLAYING WITH FALSE STATEMENTS

Now, what *was* that definition of x? "The length" is what was written above, as if "the length" of a garden with the given description necessarily existed, or was unique. But this is exactly what we are trying to discover: Is there such a length? Maybe there isn't. For example, one might ask for the length of the side of a rectangular garden whose perimeter is 100 yards and whose area is 1000 square yards. We can write equations until blue in the face; we can call its length x as above, so that 50-x is the width and $50x-x^2$ its area, but it should be plain that there is no such rectangle even before trying to solve the equation $50x-x^2=1000$. You simply can't enclose 1000 square yards in a rectangle with only a 100 yard perimeter (try a few guesses). Calling the length of such a rectangle "x" doesn't make x the name of anything real. This impossibility was undoubtedly known in ancient Babylonia, and most elaborately analyzed in geometric language by Euclid.

How can such a problem, as it eventuates in an equation, be understood in the first place, then? What right do we have to say "Let the length of the field be x," before we even know there is such an x? Without a more careful statement of what we are trying to find out and how, no amount of "subtracting the same thing from both sides" and the like will do us a bit of good, except maybe on multiple-choice exams. Both sides of *what*, for goodness sakes? An equality between symbols involving a possibly non-existent number — or maybe variable — named "x?" (In my second example here, 1000 for area and 100 for perimeter, x is a *certainly* non-existent length, or maybe variable, or place-holder, or unknown, yet it still seems to have a name, "x," and an equation to describe its properties. Are we permitted to debate the physiology of unicorns?

One reason for a more careful statement of the problem is that it will explain some of the wordless, comma-less, period-less "algebra" that appears in the middle of the typical solution. Consider: the textbook says we have an "axiom" stating that if A and B are numbers, and if A = B, and if C is some other number, than A-C = B-C, i.e. "subtracting the same number from equals yields equals." In the solution to the original rectangular garden problem above this fact was used in the following way:

From x(x-50)=600 we derived x(x-50)-600 = 0 by "subtracting 600 from both sides." Both sides of what?

An equation? Yes, the equation x(x-5) = 600. We call it an equation because it has an equals sign in the middle, but does that make it true? And, if it isn't true, does our axiom still hold? Indeed, the equation in question is usually false. When x = 14.7 it is false; when x = 435 it is false. What right have we to subtract 600 from two sides of an "equation" that is usually false, and then call the result a consequence of some axiom cribbed from Euclid?

(How that "axiom" got from Euclid into 19th century algebra books is a story of its own. Euclid in his axioms did not mean "equals" in the algebraic sense at all, but was talking about geometric figures, where by "equal" he meant "congruent" in the first instance, and then decomposable into pieces congruent piece by piece, and ultimately even more sophisticated equivalences than that. There is also the equality of *ratios* to be found in Euclid (Book V), with a definition of "ratio" hardly anyone remembers today. Furthermore, the "added to" and "subtracted from" phrases used by Euclid in his Postulates did not refer to anything numerical at all. Modern algebra textbooks tend to forget the origin of these axioms, and they list them along with corresponding rules for division and multiplication, too; something that would have been quite meaningless to Euclid, and which cannot be made to have meaning in his geometric context. [Footnote: See, e.g., Dressler and Keenan's *Integrated Mathematics*, Course 1, New York, Amsco School Publications 1980, p.108: "Postulate 7: Multiplication Property of Equality".])

In their present-day 9th Grade use, these statements may be *called* axioms, because of a long tradition culminating in the author's ignorance, but they are no more axioms than any other properties of the arithmetic operations construed as functions or operators. One might as well call an "axiom" the statement that if x(x-50) = 600 then log[x(x-50)] = log(600), or cos[x(x-50)] = cos(600). These statements are, as applied to "equality of numbers," nothing more than the recognition that taking cosines, subtracting 600, etc. are well-defined operations with unambiguous results. It isn't that two *numbers* are equal, in these applications, as that the two algebraic expressions are intended to be different names for a single number. In Euclid, "equality" denoted not a mere renaming of a number, but an equivalence between two genuinely different geometric entities.)

But this is a digression. Axiom or not, it is true that if two symbols represent the same number, subtracting 600 from each will yield two new symbols also representing the same number, i.e. the original number diminished by 600. Now let us return to the equation "x(x-5) = 600," which is almost always a false statement, and see why we have a right to subtract 600 from both sides of it and somehow use the result for a good purpose.

3. INDUCTIVE AND DEDUCTIVE REASONING

To understand all this we must return to the origins of algebra, which was brought to Europe in the Middle Ages by Arabs who themselves had been influenced by Indians, Babylonians and Greeks many centuries before that. The Greeks in the three hundred years between Socrates and Appollonius of Perga, and mainly in the unparalleled age of Plato's Academy, 2400 years ago, had perfected what is now called the synthetic method in geometry (and a bit of number theory as well), showing the world how to proceed from axioms and other known truths to more complicated statements by means of a sequence of airtight deductions, going from each known truth to the next by a step whose validity can no more be denied than the plain evidence of our senses — and even more so, in that Plato had some doubts about our senses that he did not entertain about geometry.

Most of human life goes in the other direction: we humans use experience more than logic. This use of experience we call inductive, as opposed to the deductive, or synthetic method. We see a thing happen and we look for its cause; if its apparent cause is consistent with what we see, we call that connection a *theory*. And *then* we use the connection, the theory, *as if* deductively (for we can never be as certain of our scientifically postulated causes as we are of the axioms of geometry) until or unless we find out it was wrong or not useful.

This method is certainly not Euclidean mathematics, but it is natural to mankind, and while it has led to many mistakes it has also given us science. The use of experience has been most fruitful of all, as Galileo explained, when the hypothetical "cause" is linked to observation, both past and future, and both real and imagined, by a deductive mathematical argument. Hence Galileo's insistence that experience be reducible to quantities amenable to mathematical method, to number and figure, as in Euclid.

The inventors of algebra were faced with problems that had no counterpart in Euclid's scheme. We want a rectangle whose sides do this and that; how do we find it? Can we begin with the 'known,' as in Euclid, whose assertions *begin* with a given circle, a given length, a given point, or *some* hypothesis, and go on from there? In the problem of the rectangular garden we haven't been *given* anything! We don't know the longer side, the shorter side, or even if there can be such sides! It *looks* as if we have been given an area; hmm, some "gift!" Area of what? Can there even *be* an area such as we hope to have been 'given?' Where do we begin?

Instead of starting with the known, as in Euclidean geometry, let us start with the unknown, BUT PRETEND IT IS KNOWN!

4. THE ANALYTIC METHOD: INDUCTION FOLLOWED BY DEDUCTION

It begins by guessing. Nobody can stop us from guessing, after all, and if we guess right we can easily show the answer is right, by "checking." If by some miracle I could think, "Eureka! 60 by 10 will do it!" I then could convince any child that this is surely a rectangle of the desired type. Of course, I couldn't convince anyone immediately that this is the *only* rectangle that would do it, and it is hard to see at first how one could show such a thing, but in fact algebra formalizes the "method of guessing" in such a way as to answer both sorts of questions: (1) Find a number or numbers that answer the problem, and (2) Show that these are the only answers there are.

The Arabic method of "algebra" (a word itself of Arabic origin, having to do with taking apart and putting together) is entirely systematic and convincing, but accomplished the goals (1) and (2) in the opposite order. It first finds out the only (possible) answers there are (or, rather, can be), and *then* shows them, or it, to answer the problem in fact.

Instead of starting with the known, as in Euclidean geometry, let us start with the unknown, BUT PRETEND IT IS KNOWN! What is unknown? The length of the rectangle, for one thing — and, for that matter, the very existence of such a rectangle. O.K., we pretend there is such a rectangle and that we know its length: we give it a name, "x." But remember now, x is really the *pretend* length of the *pretend* rectangle that we are *pretending* to know all about, that solves the problem, if the problem can be solved. Maybe it can't. We are not entitled yet to guarantee the problem can be solved — we have earlier, above, seen an apparently similar one, that can't be solved — but we can pretend this one has a solution.

If the pretend length is x then the pretend width is x-50; that's what the problem demands. Some children have trouble with a number like x-50, which looks more like a 'problem' than like a 'number.' We can explain, though, that this is because we don't actually know what number x is. If x were 258 then the width would be 208, which also could be written 258-50; if the length were 111 the width would be 61, which also could be written 111-50. So, if L is the length, the width is L-50. In our case we called the length x; so... "x-50" is the width. The pretend width. Then the pretend area of this pretend rectangle is the product x(x-50), which can be written in the 'expanded' form x^2-50x if we like, because that's what the distributive law says we can do with numbers — and remember, we are pretending that x and x-50 are numbers, maybe not known to us, but, we hope, known to God at least. Notice that x does not have to be called a "variable," or anything else with mystical import. It is a number — well — a pretend number.

Now if this pretend rectangle is to be a real one as demanded in the problem, it must be that its area is 600 square yards, or, to put our pretenses into an English sentence:

IF x is the length of a rectangle that can solve our problem, THEN $x^2-50x = 600$.

This is the key to the whole analytic method, and it is meaningless if it is not written (or understood) as a whole sentence, with a very strong "if" at the beginning and a very strong "then" in the middle. The mere equation, $x^2-50x = 600$, is not the statement of the problem. It is not even a restatement of the problem; it is only a part of a longer statement, the one that begins with "if" and ends with "then." In the language of English grammar, the equation "$x^2-50x = 600$" is but a clause in a complex sentence.

A clause, in English grammar, is a statement that sounds a bit like a sentence itself, since it has a subject and predicate of its own, but within a sentence it doesn't actually say what it sounds as if it is saying. In a true sentence a clause can nonetheless be false. "If pigs could fly, then they would have wings." This sentence is true, even though both its clauses happen to be false. Such is often the case with sentences of the "if... then ..." form, which is what most mathematical sentences sound like. Of course, some of the clauses might be true, too. But we must not confuse the truth of the sentence with the truth of the clauses. We can even know certain sentences to be true while we have no idea whatever whether the clauses in it are true or not. We don't even need to *care* if the clauses are true (when taken as if they were sentences of their own) or not. Try this one: "If John is 6 feet tall and Jim is 5.9 feet tall, then John is taller than Jim." Who John? Who Jim? Doesn't matter; the *sentence* is true, even though it says *nothing at all* about John or Jim, or even whether they exist.

Thus in our restatement of our problem one need not ask whether "x^2-50x = 600" is true or false. It is an *equation*, to be sure, a statement that a couple of things are equal, but, like "John is six feet tall" and "John is taller than Jim," it is just part of a true sentence, having no truth value of its own, except the knowledge that IF the opening clause or clauses are true, this one is, too.

Despite these uncertainties, we have got somewhere; we have narrowed down the problem. IF the problem can be solved, THEN x will have to satisfy the equation x^2-50x = 600. Very well; next question: Are there any numbers x which do in fact satisfy "x^2- 50x = 600?" To answer this, we go on with "if... then..." sentences.

If x^2-50x = 600 then x^2-50x-600 = 0. Why? Because x^2-50x really and truly = 600? NO! Don't let a student believe this for a minute! We don't know if that equation is true (it usually isn't, remember), or that there exists even one value of x which would make it true. What we know is that IF it were true, THEN the second statement would also be true. Subtracting 600 from a certain number, whether it is called x^2-50x or is called 600, can produce only one result, and since we happen to know the result is 0 when the "certain number" is called 600, so we also know the result is 0 when that "certain number" is called x^2-50x, *provided* x^2-50x is another name for 600.

One can say here that "the same quantity subtracted from equals produce equals," and that is a common way to remember the drill, but in logic it doesn't say very much, for x^2-50x and 600 are not just "equals" in the sense of Euclid. x^2-50x and 600 are here assumed to be the same THING, a supposedly "certain number," except that one of the descriptions of that number is more complicated than the other. OF COURSE subtracting 600 from a thing is the same as subtracting 600 from that thing! Only the names are different. And don't forget, it is only a pretend equality to begin with, in that we are *assuming* we are dealing with a number x, for the moment, that *does* make x^2-50x that real thing, 600. Who knows but that we might not someday find out that there really cannot be any such number x?

5. A CHAIN OF IMPLICATIONS WITHOUT TRUTH

Now we can apply a rule of logic called "the transitivity of implication." There was a time when textbooks made much of this idea, which is really only common sense which we use every day. The rule is this: If A implies B and if B implies C, then A implies C. What are A, B, and C here? They are not numbers, they are statements. The clause "A implies B" is mathematical shorthand for the statement "If A, then B," and it is sometimes more convenient to use the word "imply" and its allies than to go through the entire "if...then..." routine.

In the present case our statements A, B and C are as follows:

A. "x is the length of a 600 square yard rectangular field whose width is 50 yards less than its length;"
B. "x^2-50x = 600;"
C. "x^2-50x-600 = 0."

Remember, these are merely statements, clauses, things that look like assertions but are really only parts of assertions we intend to make seriously. We have already established that A implies B, though we wrote it down in the "If A, then B." format. "Subtracting 600 from both sides" is the most usual language we use to justify, in this problem, "B implies C." So the transitivity of implication, combining the two assertions, tells us "A implies C," or "If A, then C;" that is,

IF there is a rectangle answering the conditions of the problem and x is its length, THEN x^2-50x-600 = 0.

Well, now that the idea is plain, that at each step we are faced with a hypothetical statement and not an absolute statement, we can speed things up a little, making our explanations briefer. We continually use the transitivity of implication to permit us to "forget" the intermediate stages of our argument. Knowing A implies C permits us to forget all about B from now on. B has served its purpose. Similarly we will soon be able to forget C, as follows; consider the two clauses:

D. "For any number x whatsoever,
x^2-50x-600 = (x-60)(x+10);" and
E. "(x-60)(x+10) = 0."

Statement D is simply a true statement, as everyone knows and anyone can check using the elementary rules of arithmetic (the distributive law, etc.). That D is true for all real numbers x is not trivial, of course, and it demands a careful definition of "real number" before it can be asserted.

(Actually, D is true not only for real numbers, but also for complex numbers and for many things that are not numbers at all, provided addition and multiplication are suitably defined for these things among themselves and between these things and ordinary numbers. Square matrices of size 7X7 are an example, but this is by the way.)

Is E a true statement, like D? Of course not. For most values of x it is false. What is true is this: If C is true (for a certain x), then E is true. Why? Because D assures us that the left hand side of C is the same as the left-hand side of E *even though we do not know what x is*, and so if C is true, then E, known to be the same statement, is also true.

Here is where we stand now: A implies E. If there is a length x that does our job, x satisfies the equation in E. From here it is easy. The product of two numbers can be zero only if one or both of the numbers is zero. So, if E is true, then so is F:

F. "x-60 is 0 or x+10 is zero, or both."

Finally, if x-60 is true, then x = 60 (I won't repeat the details about doing the same thing to both sides), and if x+10 is true then x = -10. We can discard the "or both" because we know a single number named x cannot be *both* 60 and -10. But we do have to pay attention to the "or." In other words, F implies G, the statement

G. "x = 60 or x = -10."

Combining all the implications in a sort of chain, A implies B implies C implies E implies F implies G (remembering D was merely a truth we used along the way) we end up with the statement "A implies G" worded as follows:

If x is the length of a 600 square yard rectangular field whose width is 50 yards less than x, then x = 60 or x = -10."

We see from this statement that we do not yet have the solution, if any, of the problem; all we know is that any number which is not 60, and is not -10, will not solve the problem. This is rather a limited result, but it does clear away the underbrush. (Notice that we have now answered one of the questions about the original way I quoted a typical solution of this problem: The word is "or," not "and.") And the actual solution is now not far away. With only two possible answers, we don't have to have a flash of inspiration and shout "Eureka!" We can systematically try out the two possible solutions. Try 60: Then the width is 10, and since 60•10 is indeed 600 we have a solution. Put a circle around it. Ten points? Not yet; there might be another answer, since we haven't yet excluded -10 by all those implications. But any fool can see that -10 can't be the width of a rectangle of area 600, so we reject -10, as the book said. There is one answer, and the answer is 60.

6. CHECKING THE "SOLUTION"

This last part of the argument, the actual multiplying out of our candidate answer (x = 60) by the number fifty less than x, to see if it indeed gives us our area of 600, is called "checking the answer" in most schoolbooks, and students and sometimes teachers tend to consider this part a check on whether or not one has made a numerical error somewhere along the way. [Footnote: The Dressler and Keenan *Integrated Mathematics* mentioned earlier is but one among many texts containing no logical explanation of why one has to check an answer. Their typical instruction is "solve, and check, ..." and they make it appear that if the "solve" part contains no errors the "check" is supererogatory.] It is true, of course, that if one has made a numerical error the "checking" step will very likely uncover it, and this already makes the step valuable, but the logical function of the "check" is not often mentioned.

For another example, the book *The Teaching of Junior High School Algebra*, by David Eugene Smith and William David Reeve (Ginn & Co. 1927) was written by two of the most prominent mathematics educators of the time, both professors of the teaching of mathematics and each the author of numerous books on the subject. On page 191, a paragraph headed *The Value of Checking* contains this instruction for future teachers of algebra:

> On the whole, however, it is usually better for a pupil to solve one problem and check the result than to solve two and not check at all... (1) he does a piece of work that is ordinarily quite as good an exercise as the original solution; and (2) he has the pleasure of being certain of his result and of his mastery of the whole situation.

Smith and Reeve thus consider checking to be good for the student; what they fail to mention, and probably don't even have in mind, is that "checking" is in fact the only genuine proof of the "result" they think was already in hand.

...what they fail to mention, and probably don't even have in mind, is that "checking" is in fact the only genuine proof of the "result" they think was already in hand.

For in truth, the "result" they refer to (or the results, in the present case 60 and -10) is only *hypothetical* until the checking, the real proof, is done. Otherwise, -10 is just as good a "result," having been obtained by the same means as the 60. But 60 "checks" in the problem, while -10 — which solves the *equation*, to be sure — fails any check imaginable concerning the area of a rectangle with such a side length.

The so-called check, simple as it might appear, is really the *deductive* proof in the sense of the ancient Greeks, that our answer is right. What is a deduction? It is an argument that proceeds from something *given* to something else we then deduce from it.

In the present case we are now (after all that analysis) given a length 60 yards to study. We can actually build parallel garden walls 60 yards long and the other walls 10 yards long, i.e. fifty less than the length, and compute the area. Behold! (The word "Theorem" is ancient Greek for the English word "Behold.") Behold, the area *is* 600. No ifs or buts here. This particular "theorem" is pretty trivial, but it is a theorem nonetheless: What is this theorem? It says that if a field is 60 yards long, and 50 yards less than that in width, then its area is 600. That's all the problem asked us to show, isn't it? And in truth, we didn't really know that before we got to the so-called "check" of the answer; all we knew earlier was that IF a field did this and that, its length had to be — *if anything*! — either 60 or -10. The "check" is in fact the solution, while what is usually called the "solution" is nothing but the narrowing-down of possibilities.

Yet the traditional "solution" did tell us something else, perhaps equally valuable. It told us that the number we did check out by multiplication was the *only* one. Or the only positive one, anyhow, and since we didn't want a negative one we now know our solution is unique. There is only one set of dimensions for a garden with the properties demanded. The "theorem" given us by our *check* tells us that 60•10 worked; the other "theorem", given us by the preceding *analysis*, told us that *only* 60•10 could work, unless we wanted to get into negative "lengths," whatever that might mean.

For there does remain a nagging question about that -10. Where did it come from? Of course it can't be the solution to the problem, but it *was* a solution to the *equation* that somehow got into the problem. If you diminish -10 by 50 you get -60, and (-10)•(-60)= 600. If -10 checks in the equation, and the equation expresses the conditions of the problem, maybe there is some reason for its having turned up there. Why do we reject it? Because we know something about gardens? What have gardening facts to do with mathematics?

Suppose we hadn't been talking about gardens, but about something we didn't have so much advance information about? How would we have known to reject the "wrong" solution? How wrong is it? It checks in the equation, doesn't it?

Well, there was a slippery phrase two paragraphs back: "...the equation expresses the conditions of the problem..." That isn't quite true. The conditions of the problem were two: First that x be a positive number, since we are looking for the length of the side of a real garden, built of real fencing in a real city; and second, that the equation be satisfied. This is how we know to reject the -10. Had we been more careful in stating the problem, we might have put it thus at the very outset: "Find the (positive) *length* of the side of a garden... Then at each step of the narrowing down part of the solution, well before the "check," we would repeat "positive number" before the symbol "x," e.g. "Let x be the positive number of the pretend length of ..." and so on. We would end, "Then the positive number x must be either 60 or -10," and it is clear that our final statement would be "Then x must be 60" (if such an x exists). There would be no need to worry further about the -10, but the check that 60 works would still be needed as before.

Part II of this article will be published in the next issue of the Humanistic Mathematics Network Journal.

Ethical, Humanistic, and Artistic Mathematics

Contributed paper sessions at the Math Association meeting
January 1999 San Antonio, TX

Organizers: Robert P. Webber, Longwood College
Alvin White, Harvey Mudd College
Stefanos Gialamas, Illinois Institute of Art

Description: This session will feature talks that relate mathematics and mathematics teaching to the culture in which they are embedded. Papers discussing any of the three following themes are welcome:
* Ethical dilemmas and considerations in mathematics
* Humanistic mathematics
* Teaching mathematics to art students integrating an iconistic approach, guided inquiry, or any other philosophy or methodology

Send papers by surface mail, email, or fax to:
Professor Alvin White
Harvey Mudd College
Claremont, CA 91711
email awhite@hmc.edu
fax 909-621-8366
phone 909-621-8867

Please state which of the three themes your paper addresses.

The Poetics of $E=MC^2$

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Though I personally admire the rolling meter of "S equals R Theta" and am even perversely attracted to the stolid but dogged determination of "D equals one-half A T Squared," I concede that no other mathematical formula enjoys such eloquent poetic features as Einstein's famous "E equals M C Squared." An equation that describes the paradoxical equivalency of matter and energy, $E=MC^2$ spawned other paradoxes, including one between the apparent simplicity of the formula and the fact that it was incomprehensible to most people. Furthermore, the poetics of $E=MC^2$ render in sound the paradoxical equivalency that the equation describes mathematically.

$E=MC^2$ brims with assonance (repeated vowel sounds, in this case repeated *E*-sounds) and alliteration (repeated consonant sounds, in this case repeated *S*-sounds). In general we can associate the higher-pitched vowel sounds with energy, both because of their comparative weightlessness and because in every syllable they are required to animate the consonants, just as energy invigorates mass. A consonant string is unpronounceable without a vowel; mass is inert without energy. Moreover, vowel sounds are made in the higher regions of the throat, while consonant sounds come from farther down within the body, often from the stomach. Consonants are thus more visceral, more weighty, more mass-like. The interplay between vowels and consonants forms the basis of the aural paradox contained within the formula.

The equation's rhythmic pattern also contributes to the overall effect. Metrically, the formula splits between *E Equals* and *M C Squared*. Both halves contain three syllables. In the first half the sequential assonant *E*-sounds are stressed. Appropriate to the weightless energy it describes, the symbol *E* is voiced with a pure vowel sound; it contains no mass-like consonants and is made in the upper part of the throat. It has a comparatively high pitch.

However, the next syllable, *quals*, lowers the pitch, creates a guttural sound not shared by any other syllable, and is voiced in the middle of the throat. By dropping the stress and lowering the pitch, the unstressed, consonant-dominated quals syllable provides a brief moment of relaxation before the speaker moves to the other side of the equation. The sliding guttural action suggests transition, as the speaker becomes aware of the sound passing through the throat. Appropriately, the only "relaxed" moment in the expression appears in this "unstressed" syllable associated with equivalence. It suggests the balance and harmony implicit within the equality: the transition between matter and energy literally exists as a state of rest when we state the formula aloud.

The right side of the equation consists of three stressed syllables: M C Squared. M begins with an *E*-sound (ie.: *EM*.) Thus it raises the pitch and returns us to the *E*-sound that dominated the left side, thereby creating an aural equivalence between energy (*E*) and-matter (*EM*), as both symbols make the E vowel sound. But the *M*-sound that concludes *EM* weighs down the syllable with a lower-pitched consonant, endowing the symbol for mass with the properties of mass as well as those energy. In this way the equation's spoken poetics reflect its mathematical and physical "content," or meaning, by expressing in sound the equivalency that the formula states.

If the high *E*-sound in *EM* assigns to mass the weightlessness of energy, then the low and heavily stressed *S*-sound repeated in *C Squared* assigns the density of matter to light (C), a form of energy. At the end of the equation the pitch drops from *CEE*, which begins low but moves up to a higher *E*-sound, to *Squared*, which starts low and stays low. The *S*-sound alliteration between these highly stressed, low-pitched final syllables unites them. Moreover, the *R* and *D* sounds that conclude *Squared* firmly set the right side of the equation in the ponderous realm of mass. *C Squared* thus anchors the formula's right side in the physical and the visceral, since the sounds literally are voiced from deep within the body.

However, just as the *E* and *M* sounds in *EM* endow the symbol for mass with characteristics of both matter and energy, the *S* and *E* sounds in *CEE* give the symbol for light the weightlessness of waves and the density of particles. In this respect M and C contain within themselves the complementarily of matter and energy that Niels Bohr later described.

In conclusion, stating "E Equals M C Squared" enables the speaker to experience the equivalency described by the equation: matter is energy and energy is matter, and simultaneously energy is energy and matter is matter. By assigning to the symbol for mass an assonant *E*-sound, the "poem" endows properties of energy to the concept of mass. Conversely, by assigning the more weighty *S*, *R*, and *D*-sounds to light, the poem gives to energy the characteristics of mass In this way apparent opposites are shown to be equivalent when we speak the sounds that describe them matter is energy, and energy is matter. At the same time opposite sounds literally divide the equation into two distinct halves: a left, vowel-dominated side that sounds like weightless energy, and a right side firmly anchored by consonants in a physical presence. The assonance in the first half and the alliteration in the second suggest that energy is energy, and matter is matter.

From a poetic standpoint, if not a mathematical one, we are fortunate that Einstein presented the equation as he did, instead of as he might have: "E Equals C Squared M," "M Equals E Divided by C Squared," or, my favorite, "C Equals the Square Root of E Divided by M." Without the rhythm, assonance and alliteration of "E Equals M C Squared" the formula might never have captured the popular imagination as it has.

I LOVE TO ADD

I love to add
It makes me glad
It's easy and fun
To add one plus one

And when I'm blue
There's two plus two
It fills me with glee
To put three with three

It's never a bore
To combine four plus four
It's a slam cool jive
When five's with five

It's like a magic trick
To add six and six
I go to heaven
Each time seven's with seven

I just can't wait
To add eight plus eight
It feels so fine
To add nine plus nine

You and me is us
A plus!
So whenever I'm sad
I sit down and add

Kyle Cotler

MATH RULES

Middle school is really cool.
And math class gives us the tools
To face the world unafraid,
Because we can make the grade.
We can now multiply, divide, add, and do subtraction.
With integers, metrics, decimals, and fractions.

We can tackle word problems and
come out on top,
As our skills progress our confidence doesn't drop.

And now with algebra on our class agenda,
We have exponents and order of operations to remember.

To solve for X requires some introspective,
As all my classmates turn into detectives.
If we keep the balance in each equation,
We celebrate the auspicious occasion.

Ms. Schaeffer is very patient and always kind,
She keeps us alert as we sharpen our minds.

So when all is said and done,
I think math is very fun.

Blake Mayer

Death, Trial and Life

Prem N. Bajaj
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In Mathematics, the value of attempt is often underestimated by students. In almost all cases, struggling to find the solution is as significant as finding the solution itself. Indeed, in many cases, the student's solution turns out to be better than the solution given either by the instructor or the textbook. The following anecdote is meant to illustrate this point.

During Fall 1965 semester, a graduate student at Western Reserve University (now Case Western Reserve University), was taking a course in real analysis. The mid-term exam - indeed the first in the course - consisted of four problems carrying twenty-five points each, and the last problem, presumably difficult, had a hint provided. But deciphering the hint seemed to be as difficult as the problem itself. Being unable to give even an attempt for the problem, "death" seemed to be certain for the student for this course. Missing one problem completely out of four? Besides, a student cannot be sure to have got all other problems completely right.

With trials, made independent of the hint, the student got an idea for another approach for the solution. But having spent time on trying along the lines of the hint, the remaining time allowed the student to provide only a sketch - an overview to be exact - of the solution that he had in mind,

However, the instructor, Dr. Lazer, accepted the sketch as a complete solution. Later, Dr. Lazer filled up the details and handed out the (mimeographed) solution with the note "Idea due to XYZ" in the next class. It got life in the student and brought home the importance of one's own effort. To a beginning international student, it WAS a 'pat on the back.'

Incidentally Dr. Lazer, being both a great teacher and a successful research mathematician, was a counterexample to the usual notion "teaching or research." He seemed to believe in "teaching AND research."

TESSELLATIONS

Tesselations are quite neat
They always stick together like gum on your feet
They have no gaps or spaces
They're always on the floor
(Whether tile or wood, maybe you can think of more)
When you get a chance
You can tap dance and clap galore

They're always flat-planed surfaces
As anyone can see
Finding tessellations is as easy as one-two-three

Now that you know this information you can go
across the nation in search of tessellations!

Zan Jabara

MR. TRAPEZOID'S WALK

He walked on a diagonal
Straight through the park
As he looked up through the sky
A rainbow seemed to be making an arc
It ended at the public square
Where a circle of children were playing there
At a popcorn stand people were standing in line
Above them was a huge rectangular sign
The trees on Octagon Avenue cast a shadow most profound
It looked just like a triangle as it reflected on the ground

Michael Pillar

AM 98
UC Berkeley August 3-7

After a successful six year run, AM 98 — the premiere national Art & Mathematics Conference — is coming to UC Berkeley this August. This is the premiere national forum for bringing together mathematicians, artists, and educators to discuss topics of mutual interest — for example, visualization, symmetry, proportion, tessellation, polyhedra. The first three conference days are devoted to invited presentations; the final two days to intensive teacher workshops. As you can see from the program, (much additional information available on our web site: http://http.cs.berkeley.edu/~sequin/AM98/program.html) conference presenters include some of our most prominent artists and mathematicians.

Poincaré remarked that "a scientist worthy of the name, above all a mathematician, experiences in his work the same impressions as an artist; his pleasure is as great and of the same nature." AM98 is dedicated to helping artists, mathematicians, and educators understand and enjoy the source of that pleasure. We hope to see you there, and we hope that you will pass this information on to other interested colleagues. Thank you.

Tentative Program for ART-MATH Conference '98

Monday, Aug. 3, 1998
Nat Friedman, Howard Levine, Carlo Séquin : Welcoming Remarks.
Linda Dalrymple Henderson : Signs of The Fourth Dimension In Modern Art.
Bruce Beasley : Polyhedral Sculpture With Complex Intersections.
Ken Herrick : Fractal and Kinetic Artworks.
Harriet Brisson : Structures of Infinite Extension.
Mike Field : Designer Chaos.
David Hoffman : Minimal Surfaces.
Ad hoc presentations by participants.

Tuesday, Aug. 4, 1998
Charles 0. Perry : Implications of Mathematical Sculpture.
Carlo Séquin : Sculpture Design as a Programming Task.
Brent Collins, Steve Reinmuth : From Design to Pattern to Bronze.
Mariorie Senechal : Generalizations of Penrose Tilings.
Bill Thurston : Math Visualization.
Scott Kim : Inversions, Symmetry.
Ad hoc presentations by participants.

Wednesday, Aug. 5, 1998
Helaman Ferguson : Mathematics In Stone and Bronze.
Arthur Silverman : New Tetrahedral Sculpture
Stephanie Strickland : Poems In Conversation With Mathematics and Hypertext.
Gyongy Laky : Geometry of Form and Sculptural Constructions.
John H. Conway: On Knots and Polyhedra.
Ad hoc presentations by participants.

Thursday, Aug. 6, 1998
Kevin Lee : Tesselations.
Mike Field: PRISM - PRograms for the Interactive Study of Maps.
Nat Friedman : Fractal Stone Prints.
Howard Levine : Anamorphosis.
Ad hoc presentations by participants.

Friday, Aug. 7, 1998
George Hart : Polyhedra.
Carlo Séquin : Virtual Sculpting on a Graphics Workstation.
Karl Schaffer, Scott Kim : Geometrical Dance: Polyhedron Transformations.
Helena Verrill : Palmer's Folding Patterns.
Ad hoc presentations by participants.

Registration for AM 98

Name ____

Home Address ____

Home Phone (   ) ____

Business address ____

____

Business phone (   ) ____

Fax __________________________ E-mail ____________________________________

There will be rooms for participants to view each other's slides, videos, and notebooks as well as tables for displays. Please check 0 if you plan to show:

1) slides 0,   2) videos 0,   3) notebooks 0,   4) displays 0.

Main interests ____

____

____

The registration fee is $50/day for 1-2 days and $40/day for 3-5 days until June 1. After June 1 fees are an additional $10/day. We expect a large turnout, so it is advisable to send your registration early to reserve a space. Space may be filled by June 1. Please make checks out to "Art and Mathematics." Send to Nat Friedman, Dept. of Mathematics, University at Albany-SUNY, Albany, NY 12222.

Any suggestions, ideas, topics for panels, etc. are welcome.

Suggestions ____

____

____

Motel (name and dates) ____

The Smarandache Semantic Paradox

Anthony Begay
Navajo Community College
Lupton, AZ 86508-0199

Prove that the following sentence is a paradox:
"All is possible, the impossible too!"

Solution

If "all is possible," then the "impossible" doesn't exist, hence there is no impossible.
If "the impossible too," i.e. "the impossible is possible," then not "all is possible." Contradiction again.

The two parts "all is possible" and "the impossible too" are contradictory to each other.

Reference

Le, Charles T. "The Smarandache Class of Paradoxes," Journal of Indian Academy of Mathematics, Indore, India, Vol. 18, No. 1, 1996, pp. 53-55.

GEOMETRIC SHAPES

Shapes and sizes are very cool,
That's the reason I stay in School.
Miss Schaffer teaches us our math,
She's the one who paves our path.

We learn all shapes and sizes too,
There are so many, here's a few:
Polygon, hexagon, octagon, square.
A circle's surface is very fair.

I hope we learn about more shapes,
Maybe they're in my videotapes.
You can even make shapes out of clay,
I made a triangle just the other day.

Geometric shapes can be found everywhere,
They're something that we should share.
And now my poem comes to an end,
I hope that shapes can too be your friend.

Sam Dudley

POEM

Tessellations Tessellations they are so great
they fit together like carts and crates,
They have very many sizes and very many shapes.
With no gaps and have no flaps, you wouldn't know
why they're under the subject Math.

Natalie Kashhefi

"FUN WITH ALGEBRA!"

The world of numbers can be quite fun,
When your fear of them is overcome.

Algebra looks scary at first sight,
Solving equations with all your might.

Here are a few tricks to help you see,
Just how simple Algebra can be!
Add to one side, add to the other,
Subtract from one, and from the other.

Try to get the variable alone,
So that its true value will then be known.

Follow these rules and Algebra will be,
Just as fun for you as it is for me.

Janelle Kulik