What are Mathematical Problems?

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Introduction

The importance of problem solving in the K - 12 mathematics curriculum is well documented. One of the most recent documentations is the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM] 2000). In this publication, problem solving is listed as one of the five *Process Standards*. “Problem solving is an integral part of all mathematics learning, and so it should not be an isolated part of the mathematics program” (NCTM, 2000, p. 52). Since problem solving has been accorded such prominence, it is necessary to have an understanding of what a mathematical problem is. After all, mathematical problems existed since the time of the ancient civilizations.

Several definitions of 'problem' have been put forward; yet it seems that mathematics educators are far from agreement on the meaning of this term. Schoenfeld (1992), for example, thinks that the term 'problem' has had multiple meanings. Mathematics educators have used several terms to refer to mathematical problems. Some of these are routine, non-routine; single-step, multi-step, and real-world; textbook, non-textbook. In some cases, it is clear what is meant by some of these terms; in other cases it is not. In this article I will examine some definitions of 'mathematical problem'.

Definitions

Bruner (1961) cited the work of Weldon who claimed that one needs to consider 'troubles', 'puzzles', and 'problems' when defining a problem. A 'trouble' is a circumstance or situation which makes one upset and uncomfortable. A 'puzzle' has a nice tight form, clear structure, and a neat solution. A problem is a puzzle placed on top of a trouble. Funkhouser (1990) referred to this definition as "lighthearted", and Shulman (1985) called it his "favorite epigram." I think Bruner's citation is interesting.

According to Kantowski (1977), "An individual is faced with a *problem* when he encounters a question he cannot answer or a situation he is unable to resolve using the knowledge immediately available to him. He must then think of a way to use the information at his disposal to arrive at the goal, the solution of the problem" (p. 163). The author differentiates between a problem and an exercise. In the case of a problem, an algorithm which will lead to a solution is unavailable. In an exercise one determines the algorithm and then does the manipulation. Mervis (1978) defines a problem as "a question or condition that is difficult to deal with and has not been solved" (p. 27).

Lester (1980) says that "A problem is a situation in which an individual or group is called upon to perform a task for which there is no readily accessible algorithm which determines completely the method of solution" (Quoted in Lester, 1980, p. 287 from Lester, 1978).

Buchanan (1987) defines mathematical problems as "non-routine problems that required more than ready-to-hand procedures or algorithms in the solution process" (p. 402). McLeod (1988) defines problems as "those tasks where the solution or goal is not immediately attainable and there is no obvious algorithm for the student to use" (p. 135).

According to Blum and Niss (1991), a problem is a situation which has certain open questions that "challenge somebody intellectually who is not in immediate possession of direct methods/procedures/algorithms, etc. sufficient to answer the question" (p. 37). Thus a problem is relative to the individuals involved; that is, what is a problem for one person may be an exercise for another. For example, the task 2 + 3 may be a problem for a pre-schooler but not for a middle-schooler.

A common element in the definitions of Kantowski, Lester, Buchanan, McCleod, and Blum and Niss is that there is no known algorithm to solve a problem. The problem solver has to design a method of solution.

In *Becoming a better problem solver 1* (Ohio Department of Education, 1980), it is stated that a mathematical problem has four elements:

1. A situation which involves an initial state and a goal state.
2. The situation must involve mathematics.
3. A person must desire a solution.
4. There must be some blockage between the given and desired states (p. 5).

This definition has an affective component (the desire to find a solution) which is absent in the previous definitions.

Kilpatrick (1985) defines a problem as "a situation in which a goal is to be attained and a direct route to the goal is blocked" (p. 2). In a similar way, Mayer (1985) claims that a problem occurs when one is faced with a "given state" and one wants to attain a "goal state."

The preceding three definitions refer to initial and goal states in a problem situation. The other definitions do not refer explicitly to goals.

Polya (1985), the father of problem solving, identified two categories of problems:

1. Problems to find, the principal parts of which are the unknown, the data, and the condition.
2. Problems to prove which comprise a hypothesis and a conclusion.

Blum and Niss (1991) also identified two kinds of mathematical problems. There are applied mathematical problems in which the situation and question belong to the real-world (outside of mathematics); and there are pure mathematical problems which are embedded entirely in mathematics. These appear to be similar to Polya's categories.

Teachers' Conceptions of a Mathematical Problem

Studies have been done into teachers' conceptions of a problem. For example, Thompson (1988) found that 5 of the 16 teachers whom she studied conceived a problem as "the description of a situation involving stated quantities, followed by a question of some relationship among the quantities whose answer called for the application of one or more arithmetic operations" (p. 235). The teachers' responses implied that a problem has an answer, usually a number, and there is a unique procedure to obtain that answer. Thompson (1988) found that teachers had varying conceptions of problems. For example, some teachers gave 'story' or 'word' problems as examples of problem tasks.

Students' Conceptions of a Mathematical Problem

There were also studies into students' conceptions of a problem. For example, Frank (1988) conducted a study with 27 mathematically talented middle school students to investigate their beliefs about mathematics and how these beliefs influence their problem-solving practices. She used a questionnaire, interviews and observations. She found that students believed that mathematical problems must be solvable quickly in a few steps and that mathematical problems were routine tasks which could be done by the application of known algorithms. They perceived non-routine problems as "extra credit" tasks. Students believed that if a problem could not be solved in less than 5 to 10 minutes, either something was wrong with them or the problem. The goal of doing mathematics was to obtain "right answers." Students focused entirely on answers which to them were either completely right or completely wrong.

Spangler (1992) used open-ended questions to assess students' beliefs about mathematics and found that students do have beliefs about certain aspects of mathematics. Some of her findings concurred with those of other researchers, e.g., Frank (1988).

Spangler (1992) found that one of the common beliefs among students was that a mathematical problem has only one correct answer. Students were not prepared to accept that a problem could have different answers, all being correct. They indicated that they preferred one method to multiple methods for solving a problem because they did not have to remember much. Students admitted that they could obtain the correct answer to a problem without understanding what they were doing. Students rarely checked to see if their answers made sense in the context of the given problem. They verified their answers with the teacher or by checking the text and they are not inclined to look for multiple solutions or to generalize their results.

Mtetwa and Garofalo (1989) identified the following unhealthy beliefs which students have about mathematics and mathematical problem solving:

1. In mathematical word problems the relative size of numbers is more important than the relationships between the quantities which they represent. For example, numbers which are to be subtracted are usually close in size, and numbers which are to be divided are not close in size and are evenly divisible. They claimed that teachers and textbooks help to perpetuate these beliefs.
2. Computation problems must be solved by using a step-by-step algorithm. This is a consequence of the instructional practices of teachers.
3. Mathematics problems have only one correct answer. The consequence of such a belief is that students fail to recognize/consider/accept other valid and reasonable answers. They contend that such a belief could develop from textbook answers and classroom experiences.

There is some degree of consistency between teachers' and students' conceptions of a problem. For example, they believe that a problem has one correct answer which is usually obtained by a step-by-step procedure.

Summary

The essence of these definitions is that a problem is a task or experience which is being encountered by the individual for the very first time and, therefore, there is no known procedure for handling it. The individual has to design his/her own method of solution drawing upon the various skills, knowledge, strategies, and so forth, which have been previously learned. What the individual does in the process of working towards a solution is referred to as problem solving; so the emphasis is not on the answer but on the processes involved. From this perspective many routine word problems which appear in textbooks are mistakenly designated as problems. They are not; they are merely exercises. A problem is relative to the individual; what may constitute a problem for one person may not be a problem for another because he/she might have encountered it before. Teachers and students have similar conceptions of a problem and these conceptions are sometimes inconsistent with the literature.

REFERENCES

Blum, W. & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to the other subjects--state, trends and issues in mathematics instruction. *Educational Studies in Mathematics*, *22*, 37-68.

Bruner, J. (1961). The act of discovery. *Harvard Educational Review*, *31*, 21-32.

Buchanan, N. K. (1987). Factors contributing to mathematical problem-solving performance: An exploratory study. *Educational Studies in Mathematics*, *18*(4), 399-415.

Frank, M. L. (1988). Problem solving and mathematical beliefs. *Arithmetic Teacher*, *35*(5), 32-34.

Funkhouser, C. (1990). Mathematical problem solving: A review of the literature. (ERIC Document Reproduction Service No. ED 324 217) or SE 051 596.

Kantowski, M. G. (1977). Processes involved in mathematical problem solving. *Journal for Research in Mathematics Education*, *8*(3), 163-180.

Kilpatrick, J. (1985). A retrospective account of the past twenty-five years of research on teaching mathematical problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 1-15). Hillsdale, NJ: Lawrence Erlbaum Associates.

Lester, F. K. (1980). Research in mathematical problem solving. In R. J. Shumway (Ed.), *Research in mathematics education* (pp. 286-323). Reston, VA: NCTM.

Mayer, Richard E. (1985). Implications of cognitive psychology for instruction in mathematical problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 123-138). Hillsdale, NJ: Lawrence Erlbaum Associates.

McLeod, D. B. (1988). Research on learning and instruction in mathematics: The role of affect. In E. Fennema, T. P. Carpenter, & S. J. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 60-89). Madison, WI: Wisconsin Center for Education Research, University of Wisconsin.

Mervis, D. (1978). Problem solving and the child. *School Science and Mathematics*, *78*, 270-272.

Mtetwa, D. & Garofalo, J. (1989). Beliefs about mathematics: An overlooked aspect of student difficulties. *Academic Therapy*, *24*(5), 611-618.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

Ohio Department of Education. (1980). *Becoming a better problem solver 1 & 2*. Columbus: Ohio Department of Education.

Polya, G. (1985). *How to solve it* (2nd ed.). Princeton, NJ: Princeton University Press.

Schoenfeld, Alan. (1992). Learning to think mathematically: Problem solving, metacognition, and making sense in mathematics. In D. A. Grouws (Ed.), *Handbook of research in mathematics teaching and learning* (pp. 334-370). NY: Macmillan Publishing Co.

Shulman, L. S. (1985). On teaching problem solving and solving the problems of teaching. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 439-450). Hillsdale, NJ: Lawrence Erlbaum Associates.

Spangler, D. A. (1992). Assessing students' beliefs about mathematics. *Arithmetic Teacher*, *40*(2), 148-152.

Thompson, A. G. (1988). Learning to teach mathematical problem solving: Changes in teachers' conceptions and beliefs. In R. I. Charles & E. A. Silver (Eds.), *The teaching and assessing of mathematical problem solving*, Vol. 3 (pp. 232-243). Reston, VA: NCTM.