Humanistic Mathematics: Personal Evolution and Excavations

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Preliminary Remarks

Professionals from many different disciplines and perspectives who frequently do little more than greet each other politely, have come to appreciate, acknowledge, and even communicate with each other. Those interested in exploring a diversity of fields in relation to mathematics have set up tents around a bon fire that was lit by Alvin White's newsletter, *Humanistic Mathematics Network* of 1986 - a newsletter that officially became a journal in 1993. Fields as diverse as cognitive psychology, education, history, literature, linguistics, history, philosophy, and poetry are represented in the journal. This journal has also inspired the humanistic mathematics movement, now represented by a well attended topic group at the annual meeting of the American Mathematical Society and Mathematical Association of America. In addition, it has been acknowledged as an emerging force in a recent international handbook of mathematics education published in The Netherlands. From a personal point of view, the journal has meant a great deal, not only because of the direct impact of its articles, but more importantly because it was a contributing factor in giving me the courage to write about and integrate a variety of fields - a feat that stretched considerably the bounds of my expertise.

With much appreciation for such encouragement, I reflect in this essay on the evolution of my own writing about the concept of humanistic mathematics. I do so by setting my first publication on the topic in *bas relief* against my writing that emerged some thirty years later. I will not in this brief space (shades of Fermat and his marginal note!) have a chance to paint the variety of self-portraits that emerged over this time span. I will, however, point to a number of contributing factors that influenced the change. I propose this act of introspection as a case study of one person's struggle with new ideas.

My first article, published in 1973, that explicitly highlighted the word "humanistic" was playfully entitled "Mathematics and Humanistic Themes: *Sum* Considerations" (MHT). The evolved book, published in 2001 is less playfully entitled, *Reconstructing School Mathematics: Problems with Problems and the Real World* (RSM). As I look back at MHT, it becomes clear to me that this article planted the seeds for much of my subsequent writing. Perhaps the most dominant theme -maybe an obsession- has been a focus on problems and their educational uses. As will be obvious when I discuss some of the humanistic categories, part of that focus is ameliorative with regard to problems. That is, I point out "near relatives" of such concepts as problem solving, and indicate the educational shortsightedness of excluding them from the educational scene.

Though it may be a bit of a procrustean bed, three of the four themes that I use in comparing the evolutionary stages described in the next two sections are the same. They are: (1) world and self; (2) "why?"; and (3) problems.

An Initial Foray

MHT is set against the backdrop of two curriculum agendas of the early 1960s that influenced almost all of the school disciplines: the "structure of the disciplines" movement and "discovery learning." In mathematics, the focus at all levels from elementary school through graduate school was upon uncovering at every stage the fundamental axiomatic nature of the subject. In the elementary and secondary levels, it focused not only on deductive proof but upon having students "discover" rather than being taught explicitly what these principles were (Bruner, 1960 and Brown 1971, 1976). Apologies for the sexist language throughout my piece, but few of us knew any better in those days. There are too many male references to (sic) the reader beyond the first time it appears.

World and "Self"

The article begins with the following remarks about missing elements of "self" and how one views the world.

> If one starts with the hypothesis that aspects of mathematics learning could have an impact on the way one conducts his life (sic) and views the world in other domains as well, what are some of the possible mind-expanding ways in which it might be conceived? ... It is unquestionable that the "structure of the disciplines" movement of the past fifteen years solved some problems in the past decade for some people. ... Supposedly there was de-emphasis of such bugaboos as rote memory and drill and a renaissance of understanding and discovery.
>
> No matter how exciting and successful such an enterprise might have been... this development of the past decade and a half was bound to be uneventful from at least one point of view: impact on students as human beings living in a complicated and problematic world. The basic question asked by curriculum designers was: How might I best convey the nature of mathematics (especially the twentieth century spirit) to youngsters? A much more daring question [would] be: How might we use mathematics (among an arsenal of other things) to convey knowledge and attitudes towards the world and about oneself that would be valuable in many non-mathematical contexts [as well]? (pp. 191192).

I explored what I meant by "self" in reflecting upon the activity of milking *one* problem in considerable detail. I selected a famous arithmetic progression- the Gauss problem of finding the sum of the natural numbers from 1 to 100. The myth goes that this is a problem that Gauss' fifth grade teacher gave her students in order to be able to do some desk work, thinking that she would be able to keep her class occupied for an extended period of time. Of course, the famous Gauss got the answer in less than a minute by doing something akin to noticing that the first and last numbers added to 101, and that the second and penultimate terms added to 101, and that the third and second from last added to 101, and so forth. From this a general pattern emerges, and it of course would not be difficult to prove by induction that a good summary of that scheme would be: $(n+1) \cdot (n / 2)$

Impressive as that insight is, I chose to see the problem from a number of alternative points of view. I came up with different algebraic ways of seeing how one could generalize the solution to finding the sum of the first n natural numbers depending upon (1) whether the last number was even or odd, (2) whether each term was given its "proper" value or rather was assigned an imagined "constant phantom value" regardless of the actual value of each term. This may seem a bit cryptic without repeating the detailed analysis in the article, but the general point that I raised was the following: If we choose *n* to stand for the last number, all formulas can easily be shown to be logically equivalent to: $(n \cdot (n+1)) / 2$

" Why "?

The above encapsulation, however, does not answer the question of why anyone might expect them all to be the same (especially for the progressions that had different parity). There is a formal connection that is revealed in a deductive proof that arrives at equivalent expressions, but no amount of formal proof is persuasive in the sense that we might have anticipated that the different formulations for odd and even last numbers would be equivalent. But is there a *deep* connection that indicates a priori that the formula for the sum should be independent of parity. In the article, I explored that formulation via the following well known sketch:

$S = 1 + 2 +... + (n-1) + n$

$S = n + (n- 1) +... + 2 + 1$

$2S = (n+l) + (n+l) +... + (n+l) + (n+l)$

Thus, $S = (n, \cdot (n+l))/ 2$

As revealing as the proof is, something is still missing. This proof only further entrenches the question "why?" for though it works, we are left with the mystery of figuring out how it is (why?) anyone ever came up with the idea of writing the sum first in ascending and then in descending order as a starting point. When we think of the sum of numbers from 1 to n as a staircase of sorts, however, and see it as almost a rectangle (if only a second set of stairs were rotated and placed on top of it), we can begin to imagine how anyone would have come up with bizarre re-formulation of the problem in terms of getting twice the sum. That is, we begin to understand what motivated the selection of a peculiar algebraic trick only after we have allowed ourselves to think of the problem in geometric terms.

The geometric conception of the problem enables us to move beyond purely formal connections between odd and even progressions, and to come up with a *deep* connection. In seeking the educational potential of such categories as *formal*, *surface*, and *deep connections* among problems, however, I suggest caution in disparaging what might be considered to be shallow associations. Consider the following from MHT:

> What can we learn from this problem? Max Wertheimer and other gestalt psychologists in analyzing this problem have placed primary emphasis upon blind and deep discoveries and applaud approaches that are insightful. Thus Wertheimer finds fault with students who approach the problem of finding a desired sum by combining two sums written beneath each other and handled algebraically. He would claim that though technically correct, it is blind to inner meanings of the ideas. I think his emphasis is misguided, though he provides an intelligent first approximation for the analysis of an important educational issue. Notice that the form of the deep *answer gotten* by writing the algebraic sum twice was what might have inspired us to search for a geometric solution that *is deep* in the sense that it does not depend upon parity.
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> It is not that we ought to abhor and eliminate solutions that lack *depth in* the sense that they do not get at the inner workings of the mathematics. Instead we would like to suggest as a worthwhile educational program the explicit analysis of problems along the dimensions of surface, *deep a*ndformal similarity and differences. When is the similarity in approach only one of surface similarity (e.g. when the only connection might be that the same words are used for the problems that otherwise share no common ground)? When is the connection a *formal* one? When *deep?* More importantly how can we learn to encourage valuable thought by searching for *interaction* among and between these three categories? (p. 199).

But where does the "self" come in as we review some of these mathematical issues? Part of the answer is that comparisons along the three dimensions (surface, formal and deep) is not automatically *given,* but rather is negotiable. In discussing these categories and how they bear upon their own view of the world, students not only apply them to mathematical as well as nonmathematical experiences in the world, but can be encouraged to criticize and extend them as well. In doing so, they are engaging in a fundamental human act -that of comparing objects. The point is that any two things in the universe are the same or different *with regard to* some criteria. Furthermore, we can come to realize that frequently there is considerable ambiguity with regard to implicit criteria of comparison when we ask "why" two things are the same or different. It is sometimes a rude awakening to realize that we intended something other than we thought only when we receive an answer to a "why" that feels uncomfortable (Brown, 1982).Our future as a civilization may very well rest in our ability to see the many *with regard to* categories that are common across diverse languages and cultures. A great deal of interpersonal as well as international conflict is a result of our finding differences rather than similarities *with regard to* human qualities. Understanding the functioning of *with regard to* and the ambiguity of *why* may go a long way in affecting self-understanding. Mathematics, with its fundamental concern with what it is that ostensibly different systems share - the foundation stone of the concept of isomorphism structures - offers an interesting entrée in thinking about such issues.

Elegance

Elegance is a concept that is difficult to define, but its attributes frequently include the following: (1) an elegant solution is brief (relative to competing ones); (2) the solution involves a creation that is unexpected; (3) once created, an elegant solution is relatively easy (sometimes) to understand; (4) elegant solutions have the potential to open new ways not only of understanding a particular problem, but of creating new ones.

Many of my colleagues agree that one of the most elegant solutions they have come across is that of Euclid's proof that there are an infinite number of primes. It is unexpected in that "out of the blue" he creates a new number based upon any given number of primes (their product plus the number 1). Furthermore though it is not known whether that new number is itself prime or composite, Euclid's analysis reveals that it makes no difference in terms of the large picture- that one new prime number must exist. The proof is easy to follow (and even to complete once the number is presented on a platter), but ironically is a bit slippery for many people first seeing it to not appreciate that it is irrelevant whether or not the new number is prime. It is not hard to appreciate that his proof also raises a number of other questions to explore.

In MHT, I began to explore an aspect of elegance that I had not appreciated before. That is, it is one thing to come up with a definition of elegance. It is another, however to lay out the *terrain* within which elegance might reside. Thus:

> It should be obvious in the example of the infinitude of primes that we have been referring to elegance with regard to *one* aspect of mathematics, and the one that is most widely associated with mathematics - namely proof. ... Where else might we look for elegance in mathematics? We might look ... at the *statements of* theorems (or conjectures for that matter) in mathematics. What are some assertions that are elegant ones? One that comes to mind is the Fundamental Theorem of Integral Calculus. Here we are told that in order to evaluate a particular function - the definite integral - we must consider which family of functions that function is a derivative of. That is, two functions that are conceptually distinct on the surface - the definite integral, intuitively seen as an area, and the derivative, seen as the slope of a tangent line to a curve at a point - are found to be linked in an unsuspected way. Analogous to (2) above for proof, this *statement* connects up concepts that at first glance could not look more unrelated (p. 201).

The above remarks are just a beginning. I leave it to the reader to come up with aspects of mathematical thinking other than statements and proofs about which the concept of *elegance* might apply. Armed with some of the above thinking about the concept of elegance, I decided to investigate it further with a class of my graduate students. I was in for a surprise.

Using the array of solutions to Gauss' arithmetic progression problem, I had them select what they thought were the most elegant ones. One student chose what appeared to me to be a very messy algebraic solution. When asked to describe what made it elegant, she confided that after spending a long amount of time coming up with a rather complicated way of combining odd and even terms for any progression, she wanted to find a way of honoring the *process* rather than streamlining it in favor of some final step. She wanted to find a way that would encapsulate how she had *gotten* there rather than what the actual "there" was. In some sense then, she was interested in capturing the personal *genealogy* of the problem -- how it unfolded in her own analysis.

Whether dealing with an array of problems, or of many different approaches to the same problem, as the above remarks indicate, *elegance is* a concept that lends itself to personal understanding. Here is how I described the potential to understand "self" in relation to elegance in MHT:

Of the many different approaches to [a] problem some are more plodding and some more elegant. ... If one is exposed to or encouraged to generate an abundant number *of* elegant and inelegant approaches, then new questions and categories regarding individual styles *of* operating may emerge. What are some *of* the important variables that affect the ways in which we operate with regard to an elegant vs. a plodding approach to problems that exist around us? To what extent are we affected by [feelings of] self-consciousness? [by the] clarity of problem? [by] the source of initiation of a problem [text book, teacher, another student, oneself]? (p. 212).

Problem Posing: From Problem Solving to Problem Posing

With this last array of questions, I found myself moving from elegance towards a theme that had just begun to capture my interest, that of problem posing - a theme that was to become more and more explicit in later years. I was wondering about the kinds of personal choices people can make in their exploration of mathematical idea. The idea is suggested below:

> [One] dimension of choice ... deals with the distinction between posing and solving problems. We have become accustomed to thinking of mathematics as an enterprise involving our attempts to solve, but rarely to pose problems, though in a most basic sense, it is ... impossible to solve a problem without posing some related ones along the way. Once we have gained some experience and been exposed to strategies of posing problems, how can we use [them] as a springboard for reflecting on ourselves as posers or solvers? How do we tend to pose problems? Under what circumstances do we do so? In what ways does our vacillation between solving and posing in mathematics suggest alternatives or reflect itself in the way we operate in other spheres as well? How does our ability or inability to pose problems (or our inclination/disinclination to do so) tend to define us with regard to notions such as authority (who's in charge), self respect and the like? (p. 212).

MHT summarizes briefly the bare bones of a specific problem posing strategy as well as a mind set, that we called What-If-Not. Beginning in 1969, Marion Walter and I began to explore the What-If-Not scheme in several articles. In MHT, there is a description of the strategy as well as a rationale for its having humanistic import:

> The following steps capture the essence of the What-If-Not process:
>
> - making observations of a phenomenon presented
> - drawing implications from these ... observations
> - using the phenomenon to imagine alternatives to it
> - negating some of the [assumptions] of the phenomenon
> - [asking] new questions
>
> This activity can be thought of as a means towards incorporating an abstraction that is "out there" in such a way that we begin to gain power over it and to feel that we possess it in some important sense. This kind of "tasting" activity is one that we tend to by-pass if our focus is primarily upon solving "it." Unfortunately

> however if we persist in by-passing this activity very long, we desensitize ourselves to the point that we no longer "taste" the uniqueness among phenomena, and though [we] may he able to gain answers to questions [we] become ... insensitive to what it means for something to be a problem and have even less of an understanding of what it means to have solved something (p. 209-210).

It is interesting how idiosyncratic (and hence related to "self") the very first step described above is. That is, not only do people make different observations of a problem or a situation, but they frequently do not even "see" what is supposedly there in the first place. We shall discuss this further in the next section.

I turn now to how some of these ideas were transformed over a thirty year span.

Evolution

An important question that was raised with regard to the Gauss problem was "why?" One of the most powerful human desires - to know why - is filled with ambiguity, and our obligation as individuals and as a community is as much to engage in a dialogue that enables us to figure out what we are asking when we ask the innocent sounding "why?" as it is to try to come up with answers. We have seen that "why" might be a search for logical proof, but it can seek other dimensions. It could mean, "Why did anyone come to see this problem in this way?" It could be a search for personal motivation. i.e. Why was anyone ever interested in this problem in the first place, or why should I be interested in this problem at this time in these circumstances? Am I driven by educational concerns? Pressure to conform for a variety of reasons?

"Why?": Its Relation to Mathematical Progress

There are other aspects of "why?" that transcend specific problems, and our twentieth century focus on problem solving (especially from a technological point of view) has tended to desensitize us to viewing the larger landscape. It was in the act of asking what we tend to mean when we speak of *talent* as a problem solver that I came to see problem solving in a more global light. In RSM, I comment:

> Despite the powerful tug of problem solving as the sine qua non of mathematical progress, there is a competing story to be told on the centrality of problem solving within the discipline. ... It was not ... a genius that was needed in order to prove once and for all that Euclid's fifth postulate was provable from the others. Rather what was needed was a different sort of insight. One had to see that the problem was in some sense flawed as it was stated. Furthermore entire disciplines ... are sometimes transformed not necessarily by new solutions to problems *per se,* but rather by new ways of "situating" ... already solved problems in a new environment. (p. 94)

An example on a grander scale involves the inclination to ask why we should conceive of a field as we do. The Erlanger program in Germany is a sterling example of asking such a question. Felix Klein in 1872 completely redefined what inquiry in the field of geometry was all about, not only by solving new problems in the conventional sense, but rather by viewing the entire enterprise differently - from the perspective of transformations. It is never problems and their solutions alone that enable us to classify a field of inquiry. Rather it is the human act of deciding what are pleasing, elegant, economical, challenging ways of viewing the field itself. We are never interested in solutions *per se,* but in solutions that are driven by other concerns as well.

Analogies for re-conceiving the class room that derive from this alternative view of progress and problem solving are explored in RSM:

> It is not only interesting for diagnostic purposes to find out what a student thinks a unit or a field of inquiry is about. Rather, a class record of its perception regarding the nature of a topic being studied for several weeks, months, or even an entire semester would reveal not only how students see what it is all about in some general sense, but how their viewpoint relates to the problems they entertain. Here it would be valuable to record dissenting opinions as well as popularly held points of view. It would also be enlightening to see how and why perceptions change over time. It is not that problem solving is irrelevant in defining and reconstructing what a field is all about. It is rather that a quite different *kind* of problem is addressed whenever an effort is made to make sense out of a variety of mathematical and educational experiences (p. 95).

One of the most enlightening experiences I had with regard to students' understanding of scope [of a field] occurred several years ago when I spent a semester observing a seventh-grade accelerated algebra class.

> The teacher had devoted considerable time teaching the concept of "unknown." She had done so imaginatively and was interested in having the students not only manipulate expressions, but understand what a variable was all about. After several weeks had passed, I interviewed Debby, whom the teacher described as the best student she ever had. ... I asked her : What do you expect to learn next semester when you study more algebra? I anticipated that she might talk about more complicated expressions and equations to solve, new systems of numbers, possible applications to other non-mathematical areas, possible ways to view what they had learned from a different point of view. Instead she answered, "I will finally learn what these x's and y's really are." (p. 85).

Debby obviously had an agenda that was quite different from her teacher's. She was letting me know that she hoped that someday these "unknowns" would be become "knowable." It was almost as if she was paraphrasing from the Bible-now she sees through a glass darkly; then she will see clearly. It is not necessarily that Debby had to be "set straight." After all, I am not sure she would be able to hear what the field was about from a teacher's point of view at this stage. It was, however, helpful for the teacher to know how her agenda and that of this student were in conflict. It is even conceivable that this young student is on the verge of seeing a view that would transform the field, much as Felix Klein did over a century before. It would be instructive, however for teachers at all levels to take regular snapshots of their students' perception of what the field they are studying is all about, where they think it came from and where it might be headed. After all, these sorts of global snapshots are implicitly responding to another why that is often suppressed, namely, "Why are we studying this problem? unit? course?"

Genealogy, History and Pseudo-history

We have not yet introduced the role of history as a humanistic aspect of mathematics. Surely an historical appreciation for the history of a problem, of a branch of a discipline and of the discipline itself would be an important component of a humanistic approach to any field. Doing so in such a way that it relates the discipline not only to other fields of inquiry, but more generally to the culture from which it emerged would enable us to better appreciate how the mind of a creator is connected with the civilization that spawns her (no sic).

Without such contextualizing, for example, it is hard to understand why it is that despite the fact that Euclid had a first rate appreciation for the concept of area, and could in fact tell how the areas of any two polygonal figures compare (one being less than, same, or greater than the other area), there is not one formula for the area of a plane geometric figure anywhere in Euclid's *Elements* (Heath, 1956).This is an observation that is a shocker to most people who have studied Euclidean geometry. Imagine how you would prove (as Euclid did) that two triangles on the same base with their vertices on a line parallel to the base, have the same area, if you do not know any formula for the area of a triangle (Brown, 1993).

Studying the history of mathematics, however, is not the only way of gaining an appreciation for what it is that history is about. There are indeed forms of historical thinking that have been implicit in some of our discussion of genealogy and of the different interpretations of the meaning of "why?" as well. The student who selected what I thought was an inelegant summary of her inquiry about arithmetic progressions was telling me that she craved a way of comparing where she was with where she ended up in the analysis of the problem.

The aspect of "why?" that essentially asked why anyone might have come up not only with a solution, but with a conjecture in the first place is implicitly enticed by an appreciation for history. Even when we cannot produce evidence for how it is that an idea evolved, we can ask an hypothetical historical-type question: *How might this idea have evolved?*

There are surely ways of investigating such a question that involves a great deal of information about the time and place of its origin. But it is also possible to investigate the question in such a way that each of us can use whatever resources we have at our disposal and imagine what might have been the circumstances that gave rise to the idea. I have dubbed that kind of personal inquiry "pseudo-history." In RSM, I discuss the use of pseudo-history with regard to Goldbach's famous conjecture of 1742. His conjecture asserts that any even number greater than two can be represented as the sum of two primes. Thus 12 = 5 + 7; 18 = 11 + 7; 28 = 5 + 23. Interestingly enough it took almost two centuries before this conjecture was even partially cracked-- something that was accomplished by the Russian mathematician, Schnirelman in 1931. Here is my description of pseudo-history in action:

> I imagined how [Goldbach] might have come up with [his] conjecture in the first place. ... There were many possible ways of imagining forerunners, but one that struck me as particularly revealing was the *converse* of that conjecture. That the sum of any two primes ... is even seemed so simple that almost nothing was needed to prove it. I imagined that Goldbach might have started with the simplistic converse and wondered what kinds of challenges could transform such a trivial observation into something worth thinking about (p. 197).

When hard historical data is not available, feminist scholars frequently come to understand historical questions about women by encouraging other women to reflect upon their own family experiences and their own lived experiences - a powerful form of pseudo-history. I conclude by imagining how that model might be incorporated in the classroom - shades of our earlier discussion of "why?"

> Pseudo-history has become a commonplace in many fields that appear to be lacking in documentation of the past. Some feminists for example, have found it helpful in uncovering a sense of womanhood, to create a tapestry of their own history based upon threads that are personal rather than derived from scholarly sources. In that case, many women find it helpful to look inward as a community in order try to understand how women experience the world and how they learned to define themselves. This kind of activity can profitably engage students in all fields of inquiry. Borrowing from the feminist model, the classroom of the present can be a laboratory for history. That is, students can learn to keep a record of how their thinking/feeling evolves over time. What is real then is not what history as an official document stipulates, but rather how we create it from within. It is a pretend history of sorts, but nothing could feel more real than to act as if we had created the history based upon our own lives. In such an environment, students could not only do explorations and reflect on them, but

> this activity could be part of the historical record of the course itself. A significant part of the curriculum would not only deal with problem solving, heuristics of problem solving and looking back at what they "discovered," but would help students recall the debates and differences of opinion that were expressed over a protracted period of time as the students themselves struggled to make sense of new ideas (p. 197).

Problem Posing: From Problem Solving, to Problem Posing, to Problem

Two major themes that propelled RSM are described in the two phrases following the colon (*Problems with Problems and the Real World*). The first is an effort to explore the traditional role of problems in mathematics education and to seek alternatives that have greater humanistic import. The second deals with our conception of mathematics and the real world. We turn first to the issue of problem.

In MHT, I had begun to talk about problem posing as an integral component of problem solving. This is a theme that was developed in several essays and in two books that Marion Walter and I wrote on the topic: *The Art of Problem Posing,* and an associated collection of essays written by many of our colleagues, entitled, *Problem Posing: Reflections and Applications.* There we laid out the strategy for the What-If-Not scheme in considerable detail and further spoke of the centrality of problem posing in mathematical thinking. In RSM, I further elaborated upon the rationale for problem posing - speaking of its philosophical basis in the act of thinking, its educational relevance, and further elaborating its relationship to problem solving.

There, however, I took a careful look at some foundational issues. These issues were motivated when I was surprised to discover answers to the following two questions by experienced teachers: (1) what are some good reasons for including problem solving in the curriculum?; (2) what are some good reasons for including problems in the curriculum? The reader might wish to try out two questions and compare their answers with a group of students at almost any level. The specific answers are not so critical for the purpose of raising these foundational questions. What is critical, however, is that the answers given are essentially the same regardless of which of the two questions is asked.

What I saw was that teachers were essentially establishing an educational agenda - that of encouraging and enabling their own students to associate problems with problem solving. I wondered why this happened and began to explore the philosophical literature on the concept of problem itself. I was particularly struck by the fact that though there was a great deal of analysis of problem solving, especially in relation to thinking, there was very little in the analysis of the concept of problem itself. I summarized some interesting discussion from a number of people from twentieth century philosophical camps as follows:

> Though there are a number of alternative conceptions of *problem* and though there are many different conditions that seem to be part of its definition, the driving logical/ philosophical force is captured most sharply by Nickles. Recall that according to him, "a problem consists of all the conditions or constraints on the solution plus the demand that the solution be found." It is Nickles' clear statement of the *demand* condition linked with the desire for a solution, that contributes to the educational fallacy (p. 77).

So we have the concept not only of *solution,* but of *demand* built into the concept of problem. I was drawn to the realization, however, that the *definition* of a phenomenon (even if appreciated implicitly) does not establish its *use.* Thus problems in educational settings need not accord with the definition of problem. If a knife is defined as a sharp object with certain properties, that does not mean that it cannot be used as a paper weight, weight lifting object, magnetic attractor, art object, and so forth.

This led me to define what I called a meta-fallacy and an educational fallacy.

> Meta Fallacy: *Once we are clear about what we mean by problem, and how it relates to constraints, demand and solution, then an educational agenda flows from that conception* (p.78).

A clearer statement of what is involved in the fallacious leap is:

> Educational Fallacy: *Given any problem, in order for it to be used for educational purposes, it is necessary that the goal ultimately be directed towards a possibly wide variety of efforts of students to solve that problem (p. 78)*

An attempt to confront these fallacies resulted in a full blown analysis of the various uses of problems, problem posing and problems solving. In particular, I explored how it is that problems could be a source for the creation not only of other problems (as in problem posing) but of situations (demand withdrawn or made more implicit) and vice versa. I comment on the educational value of the transformation of problem to situation as follows:

> To the extent that [a] question asked in [a problem creates] an insurmountable difficulty for many students, a program designed to "neutralize" problems so that they may become situations without a built-in demand might very well enable [them] to come to an understanding of "what the problem is." For many people, just removing the question at the end of [a problem] would transform a problem into a situation. For others, there may be a number of problems that remain that are implied in the description even if there is no explicit demand (p. 91).

Consider now the possible humanistic import of the reverberation of problem and situation:

> As we move from the focus of creating problems from situations, and vice versa, to that of trying to understand what motivates some people to create some situations and others to create other situations or to why some kinds of problems or situations created are more appealing to some and not to others, we begin to participate in a dialogue that enables people to reflect on what they value and how they think. When we communicate with others about what we believe can be transformed from a problem to a situation, we tell each other what we "see" even when we may not state it explicitly. That most people do not notice that in defining prime numbers, for example, the domain of natural numbers is seen as salient (and thus capable of being transformed to some other set), tells us what they take for granted. Such an observation is an invitation to try to figure out what it is in our own lives that we take for granted and "do not see." It becomes a significant question then to figure out what it might be that enables us to "see" better or differently. There are surely occasions upon which tragic events can become the impetus to enable us to see what was not there before. It is possible to view tragedy as a form of What-If-Not. It represents the modification of a situation in such a way that we see the situation itself differently (p. 91, 92).

Of course this is only a first step in exploring the educational potential of problems. In addition to finding ways of "neutralizing" problems into situations or vice versa, both mathematical and personal insights ensue when many other questions are considered, such as: Why am I being presented with this problem at this time? How does this problem relate to others like it that I have seen? Am I interested in pursuing this problem? Are there others who have a different interest in this problem than I? Why? Do I understand this problem? What would it take for me to understand it better? Differently? What would it take to confuse me about this problem?

Even when a solution is presented together with a *problem,* that pair can be a force to generate questions that have a humanistic focus beyond the solution itself. Here is some inquiry of that sort that I followed Euclid's proof of an infinite number of primes:

- There obviously is a difference between coming up with a solution and being able to appreciate it once it is presented. Why should it have been so hard to come up with the above solution when it is so brief and relatively easy to follow?
- Are there circumstances in each of our (non-mathematical) lives that exemplify the above principle: something hard to come up with but relatively simple and easy to appreciate once we have been made aware of it?
- Frequently people experience mathematics as "plodding" and "obtuse." Have you come upon mathematical ideas other than this one that relate in an intuitive way to the concept of infinity. Might some of them fit the category of "elegance"? (Think about when you were very young and falling asleep … worrying about whether or not you might fall off the end of the universe.)
- There is high irony in this solution from at least one point of view. In an effort to create this number K_5, Euclid first produced a number that could not be less prime. That is before adding the number 1 at the end, he created
  a) What might have been going on in Euclid's head that got him to make that move? Notice that he then added 1. Could he have added 2? 3?
  b) We are thinking of a temporal order of sorts in the creation of the number $(p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5) + 1$. Do you think that is a legitimate way to think of anumber? More precisely, we can think of $(p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5)$ as being the first move and adding 1 to be the second move. From what points of view does this temporal concept make sense? What alternative ways are there of thinking of the number K_5? (again realizing that 5 could be replaced by any value.)
- A lot of people who see this proof for the first time think that the argument is that either i) is the case or ii) is the case. T
- Before Euclid came up with the proof that there exists an infinite number of primes, you can imagine that some people (maybe even Euclid) were in doubt about the answer to the question "How many primes are there"? Can you put yourself in a position of such doubt?
  a) How would you create such doubt for yourself?
  b) Suppose there in fact were a finite number of prime numbers. What might the consequences be, and consequence

We turn now to the second theme of RSM- the relationship of mathematics to the real world.

World and "Self"

We arrive full circle to the first topic that we mentioned in discussing MHT. School curriculum is now more inclined to relate mathematics to the real world than has heretofore been the case. Furthermore, that connection is made not only in scientific fields, but in areas that are more closely aligned with the humanities. Here, for example, is a list of suggested topics mentioned in RSM for relating mathematics to the real world. It is found in a1989 document of the National Council of Teachers of Mathematics**,** *Curriculum and Evaluation Standards for School Mathematics:*

- *Art:* the use of symmetry, perspective, spatial representations, and patterns (including fractals) to create original artistic works.
- *Biology:* the use of scaling to identify limiting factors on the growth of various organisms.
- *Business:* the optimization of a communication network.
- *Industrial arts:* the use of mathematics-based computer-aided design in producing scale drawings or models of three-dimensional objects such as houses.
- *Medicine:* modeling an inoculation plan to eliminate an infectious disease.
- *Physics*: the use of vectors to address problems involving forces. (p. 149)

In that same document, they give the following as a secondary school example of applying mathematics to the real world.

> Suppose Anne tells you that under her old method of shooting free throws in basketball, her average was 60%. Using a new method of shooting, she scored 9 out of her first 10 throws. Should she conclude that the new method really is better than the old method? (p. 172).

RSM argues that such a conception of application is an interesting one, and in fact is more enticing than many of the contrived "word problems" that have for decades been the hall-mark of relating mathematics to the real world. Nevertheless, it is a truncated and skewed vision of the connection between mathematics and the real world. Focusing on the dominant use of mathematics as a *model* of a simplified "real world" situation, I comment on what is missing as follows:

> There is nothing inherently wrong with the notion of a model as far as it goes. In fact, much progress in the "real world" is achieved by so translating real-world problems. What is missing from an educational perspective, however, is that such a notion of application tends to isolate mathematics. That is, mathematics is frequently only a partial solution to real-world problems. Frequently one can go only so far with a mathematical analysis, and then it is necessary to introduce other dimensions of human thought and experience in order to think more about the problem (p. 16).

RSM criticizes the world view that limits the connection to being essentially that of a model. I suggest the following modification of the basketball problem:

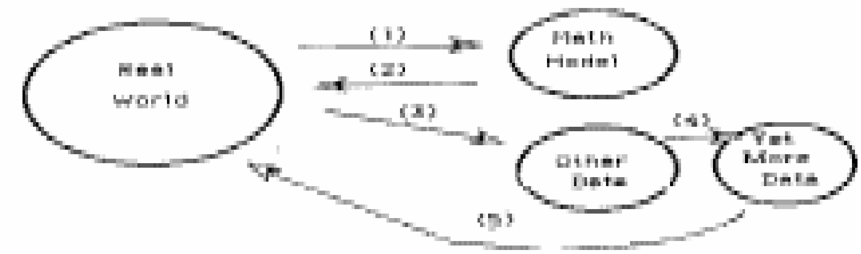
> A close relative of yours has been hit by an automobile. He has been unconscious for one month. The doctors have told you that unless he is operated upon, he will live but will most likely be comatose for the rest of his life. They can perform an operation which, if successful, would restore his consciousness. They have performed ten such operations in the past and have been successful in only two cases. In the other eight, the patient died within a week. What counsel would you give the doctors? (p. 142)

What do these two problems have in common? Though they have some elements in common (e.g. probability applied to supposedly real world problems), the medical one goes beyond the need to reach a conclusion, but rather requires a decision. Furthermore, the medical rendition deals not only with mathematical problem solving, but with ethics as well. Here, we have to consider not only issues of probability, but rather we would be inclined to support an event of low probability. Not only do we need to know what the chances are that the person will survive, but we need to address the more salient question of what kind of life is worth living.

But there are more important aspects that are missing when we make use of a "model mentality." Unlike the example of Anne and the basketball hoops, we cannot make an intelligent decision about what to do if our primary focus is on excluding irrelevancies. Rather, we need more rather than less information about our comatose friend. Here are some information we might wish to seek.

> We might want to know what sort of living will, if any, the person left. We might want to seek additional medical opinions rather than accept the conclusion about what might or might not be the result of performing the operation. In short, in order to make a wise decision, having a lot more information would be more desirable than streamlining the real-world problem for the purpose of creating a workable mathematical model (pp. 142-3).

In such real life situations, we thus relate mathematics to other fields of inquiry in a variety of ways other than that of using one as a model for another. The following suggests that we supplement the traditional scheme of model with the following kind of variation:



This elaboration of model makes clear that rather than seeking to eliminating "garbage," we need rather to seek out new data (and incorporate them with values) in order to make wise real world decisions. Eventually, I end up supplanting the notion of model altogether in an attempt to seek commonalty between mathematics and other fields. I comment in RSM:

> A substantially different point of view is revealed as soon as we begin to relinquish a hold on mathematics that is rooted in a desire to see the field as totally different from other experiences in the world-as the deductive science par excellence, the science of necessary conclusions, or the field in which knowledge that is certain may be purchased to rejuvenate some other real-world phenomenon. With this acquiescence, we invite a point of view that asks a different sort of question. What is there in the way of thinking, experiencing, feeling that mathematics shares with other ways of experiencing the world? (p. 145).

Using this alternative view of connection, we are drawn to seek connections between mathematics and language, poetry, humor, morality, confusion and so forth. In RSM I explore questions like: How is mathematics like language? (Hersh, 1997; Brown, 1997). In what ways is mathematics itself a language? What is there that mathematics shares with poetry? With humor? Is mathematics itself humorous? How so? What does morality have to do with mathematics?

While it is not possible to review the array of issues that derive from a more robust and less elitist view of the nature of mathematics, we can offer one example: that of humor. In the section on genealogy above, we mentioned Goldbach's conjecture of 1742, and the fact that the first crack was made by Schnirelman almost two centuries later. Recall that Goldbach's conjecture is that any even number greater than 2 can be represented as the sum of *a pair of* primes. What might represent a fanny first crack in solving that problem? One line of thought might be to focus on the *number* of natural numbers that have to be added in order to produce a sum that is prime. For small numbers we can actually verify that a pair can always be produced. For large numbers, the issue is more difficult. Can you come up with a pair for any even number greater than a million? While computer programming surely makes this task easier in a trial and error fashion, what do we do when we come up with an even number that is so large that we exceed the capacity of the computer?

In pre-computer days, Schnirelman actually came up with a deductive proof that told us that no matter how large the even number was, it could be represented as the sum of a certain number of primes. Given that the conjecture seeks two as the answer, what would be a funny number of primes that Schnirelman could guarantee would do the trick?

> Perhaps ten would be funny. Maybe 100, or perhaps 1000 would create a chuckle. Would the need to add 300,000 natural numbers instead of a pair create

> a belly laugh? That was Schnirelman's contribution to the problem. He showed that given any even number, we can find at most 300,000 primes that must be added in order to achieve it (p. 181).

Why is 300,000 a funny answer? The incongruity between what is needed in most practical situations and what is considered to be appealing in mathematical discourse is frequently a source of high comedy. That sense of comedy is further exasperated by the fact that many mathematical proofs are existential in nature. As is the case with Schnirelman's proof, he offered no way of actually producing the primes needed for any even number; he only showed us that they must exist (at the peril of an internal inconsistency).

In addition to asking how mathematics itself can be humorous, I look at some of the features of humor that are shared with at least some aspects of mathematics. Of the many possible commonalties, we mentioned one earlier. The element of surprise as a quality of elegance in mathematics is a central feature that both fields have in humor.

Coda (but not some running no's)

More than any other discipline, mathematics has been dominated by mischievous epistemological views. The field is popularly portrayed as impervious to doubt, scornful of the realization that the selection of what is considered relevant knowledge is a social construction, reluctant to engage in controversy or debate, disinclined to admit non-linear unraveling of ideas, and slow in acknowledging that cognition and emotionality are inextricably intertwined. Though at the time that I began writing about mathematics as a humanistic field, I was aware of the philosophical work of Gödel, and later about the contributions of Lakatos, it is only within the past decade or so that I have sought to connect those philosophical themes with issues of an educational nature (Brown, 1996a; Davis, 2001; Hofstadter, 1979; Lakatos, 1976; Nagel & Newman, 1987). If we think of unsolvability as an unwelcome guest of at least some of mathematics and if we believe that at least some of mathematics has problems that are virtually impossible to state with any degree of accuracy, (and that proof must yield to the primacy of counter-examples) then what sorts of text and what sorts of conversation are compatible with those views?

This is a hard question to answer, partly because it is, appropriately enough, a hard question to frame. In recent year, I have come upon two genres within which to express the pedagogical counterparts of these epistemological mathematical themes: the novel and the Talmud.

I came upon these genres when thinking about what implicit messages inhere in most texts. Though some of these messages are the result of poor writing technique, many seem to come close to being built into the genre itself RSM, in offering a slightly exaggerated account of texts, suggests valuable alternatives.

> Sequentiality and careful logical development are a text's hallmarks, [as is] a general lackluster "story line." It is rare, for example, that readers are encouraged to figure out the relative importance of one idea in relation to others. Furthermore, it is usually assumed that the purpose of the text is to familiarize students with information or skills about which they supposedly know little or nothing. It is not only that they lack language to describe what will unfold, but rather that they supposedly possess no previous independent experience to enable them to appreciate the topic [in some vague or imprecise way]. There is also the implication that it is the text's responsibility is to be unconfusing and to provide clear explanations. Behind this assumption is that the ideas presented not only are uncontroversial, but that they were born full blown with no labor pains to boot. In short, everything, including problems to be worked on, is "given" with

> little context for its evolution and little awareness that much is contestable. Most textbooks, especially in mathematics at the post elementary school level, convey little in the way of conflict, unsolvability, drama and emotionality. (p. 203)

In seeking the novel as an alternative genre, I was influenced by some powerful philosophical analysis of story telling as a missing ingredient in the field of philosophy itself. Martha Nussbaum (1990), for example comments,

> How should one write, what words should one select, what forms and structures and organization, if one is pursuing understanding? ... Sometimes this is taken to be a trivial and uninteresting question. I ... claim that it is not. Style itself makes its claims, expresses its own sense of what matters. Literary form is not separable from philosophical content, but is, itself, a part of content-an integral part, then, of the search for and the statement of truth. ... But this suggests, too, that there may be some views of the world and how one should live in it-views, especially, that emphasize the world's surprising variety, its complexity and mysteriousness, its flawed and imperfect beauty--that cannot be fully and adequately stated in the language of conventional philosophical prose, a style remarkably flat and lacking in wonder --but only in a language and in forms themselves more complex, more allusive, more attentive to particulars (pp. 3-4)

Robert Nozick (1990) points out that some literary figures have a reality about them even though they never existed and could not be touched, smelled or seen. Interestingly enough, the same can be said about mathematics. Who of us has not been enchanted by the "reality" of an infinite set that shares those qualities? I thus suggested that despite its traditional sense of reality, we have all had visions in our youth of reaching the end of the universe and wondering if we would fall off or if we would continue forever.

Armed with some of these insights, I wrote a mathematical novel, appropriately entitled, *Posing Mathematically (I* 996b). It is about two teachers who come upon some unexpected mathematical and pedagogical ideas, and who decide to collaborate and to try to find a forum for their ideas. They encounter criticism for both the mathematics and pedagogy by colleagues and students. After considering alternative paths towards popularizing their ideas - such as applying for a National Science Foundation Grant, or giving talks at national meetings, they choose (in self-referential fashion) to write a novel!

Motivated by some of the above criticism of text, I have found the Talmudic format to be an engaging source for the exploration of mathematical ideas. The Talmud is a sacred text and considered to be second only to the Bible in Jewish tradition. It actually consists of two main texts: The Mishna and the Gemora. The Mishna was produced in the second century A. D. and is an attempt to codify traditions, especially in relation to the Bible. The Gemora, produced in the fifth and sixth centuries is commentary on the Mishna.

What is appealing about the Talmud is both its format and style of exposition. Some people have likened this two thousand-year-old tradition to the Internet. Rosen (1998, 2000) claims that "[The Talmud] bears a certain uncanny resemblance to a home page on the Internet, where nothing is whole in itself but where icons and text-boxes are doorways through which visitors pass into an infinity of cross-referenced texts and conversations" (1998, pp. 8-9).

One of the most appealing aspects of the Talmud is that it combines careful descriptions with non-linearity. As a matter of fact, it is assumed as soon as anyone begins to read any section of the Talmud, that she has already read every other section (Lukinsky, 1987).

Each page is surrounded by commentary in the margins made by scholars conversing with each other over a period of hundreds of years. It is impossible to study any page of the Talmud without hearing the voices of generations past and without feeling an intimate connection with those rich traditions and debates. In addition to format, the style varies throughout. What begins with in an expository/didactic style soon becomes a story. In the appendix to RSM, I begin with a fanciful story that involves payment for corn seeds with calculation done in the sand. One of the protagonists calculates two numbers ending in zero by ignoring the O's and then appending them at the end. This is first (in the Mishna) presented as a non-controversial rule, but it is eventually (in the Gemora and commentary) viewed as problematic. Eventually there is a hint of the connection between this short-cut and the distributive principle. But no sooner is this done, then a story unfolds as described below:

> The Gemora stops its logical development and "out of the blue" begins to talk of a dream of one of the narrators whose daughter is pregnant (interpretive mode). He wonders about the gender of his unborn grandchild. He is then whisked away in a chariot and visits a different land that is in every way like his own, except for the fact that time and space are contracted. Everyone is considerably smaller, and events are played out in a fraction of the time it would take in his "real world." Without amniocentesis, he is thus able to discover in a few seconds the gender of his unborn grandchild (221).

As the text continues, commentary in the margin begins to explore the connections between the rather simpleminded short-cut of adding numbers ending in zero and the narrator's concern over the gender of his forthcoming grandchild. Eventually, a connection is made between the "real world" of the story and the arithmetic short-cut. The dream world is to the "real world" as "lopping off" the zeros is to including them in the final calculation. Both realms depict the concept of an isomorphic structure, though it is explored and revealed without fanfare and without making use of technical machinery that usually introduces and obfuscates what that concept is all about. In the right-hand R/D sketch, information about the "real world" (R) is mapped onto the imaginary one (D) and then back to the real world in order to determine nine months early what the gender of the child will be. Here the operation in the real world as well as the imaginary one is one of mating (m). In the case of the left hand R/D sketch, the operation is that of addition, and the lopping off of numbers ending zero (in the R world) is depicted by movement to the D world and then back again to R. In the arithmetic example, it is possible to think of the distributive law as justifying the fanciful "lopping off" algorithm [(47 * 10) + (22 * 10) = (47 + 22)* 10] depicted in the sketch below if the mapping from R to D is from n to (1 / 10) * n for any n in R that ends in 0.

Thus, instead of performing step 5 immediately, it is possible in both cases to follow steps 1 -> 2 -> 3 -> 4.

All of this needs cleaning up if we are worried about precision; also the author might be in need of some degree of psychoanalysis for invoking a Mack Truck to pull a kiddy car (when lopping off the zeros might be explainable in simpler terms), as well as for seeking analogies with such powerful sexually charged imagery. I have found, however, that encouraging students to come up with such personal analogies, and to compare the images of their real world and emerging mathematical thinking with other students has enormous power. Comparing crude sketching of the sort invoked below enables them to come to understand the quite different intellectual functions served by playful imagery of a personal nature and formal/technical machinery. It has enabled many of them to see proof as just one element of the mathematical landscape, and to define what some of the other frequently suppressed ones might be.

I have taught several courses in which students reacted to and then created Talmudic text around mathematical ideas of their own choosing. (RSM and elaborated in Brown, 2002). Below is a comment made by one of my students in a secular Talmud course. He was a private person, who had just begun to teach, loved nothing more than solving mathematical problems that were "given" and who, before engaging in the Talmudic experience, detested the prospect of writing essays -especially ones that were to be introspective. I offer this reaction not for self-serving purposes, but to sensitize my colleagues to the frequently hidden desire that many of our students have to see themselves in personal terms in relation to subject matter even when they would have us believe otherwise.

> This is truly a sad time. Yet still a time to rejoice. The semester is over. Yet so much has to be done. I am trying to think back to what I have immediately learned , and accepting that in the future, I will inevitably see more. One of the most peculiar things I have begun to develop, and which I credit this class. is a better sense of humor and love of life in general. I had mentioned to Dr. Brown about laughing until crying and bodily convulsions set in. Some of the critical analysis skills I have picked up in this course have opened up these venues of humor. This course is titled for mathematics education, and I feel I have learned a lot of ways to improve upon my mathematics teaching. But I feel that most of what I learned was how to be a better person. ... I feel that what this course has done is keep the playful spirit of the child in our education, and reminded us to keep it in our classrooms and our lives (Brown, 2002, p. 237).

The Talmudic experience invited students to uncover multiple perspectives on a variety of issues, to experience intense deliberation without being driven by the need to be right, to see story-telling as well as explaining as an important element of education, and to be tolerant of delayed or non-existent resolution. Most importantly, they began to listen to new and previously hidden voices within each of them and to engage in the risky behavior of uncovering and incorporating aspects of their lives that had previously been seen as isolated and educationally irrelevant. In short, they began to see mathematics, themselves and education in a more humanistic light.

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